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UPPER AND LOWER FUNCTIONS FOR DIFFUSION PROCESSES

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The paper is concerned with the study of upper and lower functions for diffusion processes described by the non-linear time-dependent stochastic differential equation.

1. INTRODUCTION

Let $X(t)$ be the solution of the non-linear time dependents tochastic differential equation

$$dX(t) = a(X(t), t)\, dt + dW(t) \qquad \text{...(1.1)}$$

with the initial condition

$$X(0) = X_0 \qquad \text{...(1.2)}$$

where $W(t)$ is a standard Wiener process and X_0 is independent of $F\{W(t),\ t \geqslant 0\}$ with $E\left(X_0^2\right) < \infty$.

Let $A > 0$. Let H_A be the classof non-negative, non-decreasing functions defined on $[A, \infty)$ which increase to ∞ with t.

For $h(t) \in H_A$, we say that $h(t)$ belongs to the upper class or lower class according as

$$P\{X(t) > h(t) \text{ i. o. as } t \to \infty\} = 0 \text{ or } 1.$$

In this paper our aim is to develop the integral test criterion for the solution process of (1.1) to decide whether $h(t)$ belongs to the upper class or lower class.

We give preliminary lemmas in section 2. In section 3, Theorem 3.1 gives a result analogous to Strassen's invariance principle. The integral test criterion for diffusion processes described by equation (1.1) have been developed in Theorem 3.2.

Problems of the above type have been considered by many authors. Let $\{X_n\}$ be a sequence of independent random variables. Feller[2] and Chung[1] have studied the asymptotic growth rates of $S_n = \sum_{i=1}^{n} X_i$ and $M_n = \max_{1 \leqslant i \leqslant n} |S_i|$ respectively, which are

considered to be fundamental papers in this area. In case of Brownian motion $\{W(t), t \geqslant 0\}$, Kolmogorov has developed an integral test for non-decreasing function $h(t)$ so that $h(t)$ belongs to upper class or lower class according as the integral converges or diverges. The same problem has been considered by Strasson[9] for a martingale difference sequence $\{Y_n\}$ and by Jain *et al.*[6] for the partial sum $S_n = \sum_{i=1}^{n} Y_i$. Jain and Taylor[7] have studied the asymptotic growth rate of $M'(t) = \max_{0 \leqslant u \leqslant t} |W(u)|$. For the first time Mishra and Acharya[8] have developed the integral test criterion to decide whether $h(t)$ belongs to the upper class or lower class for diffusion processes described by the homogeneous stochastic differential equations of the Ito type.

2. Notations and Preliminaries

This section is devoted to the background materials which have been used in this paper. Let us consider the stochastic differential equation (1.1) where $W(t)$ is a standard Wiener process. We assume that $a(x, t)$ is realvalued, well defined, measurable for $x \in (-\infty, \infty)$ and satisfy the following conditions,

(A_1) for some constant K and $0 < \delta < 1$.

$$| a(x, t) | \leqslant K/(1 + | x |)^{1+\delta}$$

(A_2) for $c > 0$ and x, y in $(-\infty, \infty)$, there exists a constant L_c such that

$$| a(x, t) - a(y, t) | \leqslant L_c | x - y |$$

where $| x | \leqslant c$ and $| y | \leqslant c$.

Under conditions (A_1) and (A_2) and the initial condition $X(0) = X_0$, Gikhman and Skorohod[4] have shown that, there exists a unique solution $X(t)$ of (1.1) in an arbitrary time interval $[0, T]$ and

$$X(t) = X_0 + \int_0^t a(X(s), s)\, ds + W(t). \qquad \text{...(2.1)}$$

Moreover $X(t)$ is a Markov process whose transition probability is given by

$$P(X_0, t, A) = P_{X_0}(X(t) \in A). \qquad \text{...(2.2)}$$

To prove the main theorem we need the following lemmas.

Lemma 1—Under condition (A_1)

$$E \left| \int_0^t a(X(s), s)\, ds \right|^2 = O(t^{1-\delta}).$$

For the proof of this Lemma refer to Friedman[3] (p. 184).

Lemma 2—Let $\phi\ (t)$ increase monotonically to infinity with t and $\{W\ t),\ t \geqslant 0\}$ be a Brownian motion process. Then

$$P\ \{W(t) > t^{1/2}\ \phi\ (t) \text{ i. o. as } t \to \infty\} = 0 \text{ or } 1.$$

according as

$$I\ (\phi) = \int_1^\infty \frac{\phi\ (t)}{t}\ e^{-\phi^2(t)/2}\ dt$$

is convergent or divergent.

The above result is due to Kolmogorov [see Ito and McKean[5], p. 165].

Lemma 3—Let g be an eventually non-increasing function from $[0, \infty)$ to $[0, \infty)$ and ψ be a measurable function from $[A, \infty)$ to $[0, \infty)$, for some fixed $A > 0$. For $h \in H_A$, define

$$F\ (h) = \int_A^\infty g\ (h\ (t))\ \psi\ (t)\ dt$$

which may be either finite or infinite. Assume that

(a_1) for every $h \in H_A$ and for every B such that

$$B > A > 0,\ \int_A^B g\ (h\ (t))\ \psi\ (t)\ dt < \infty.$$

(a_2) There exists h_1, h_2, two members of H_A, such that

$h_1 \leqslant h_2$, $F\ (h_2) < \infty$, while $F\ (h_1) = \infty$ and

$$\lim_{B \to \infty}\ g\ (h_1\ (B)) \int_A^B\ \psi\ (t)\ dt = \infty.$$

Define

$$\hat{h} = \min\ [\max\ (h, h_1), h_2].$$

Then for $h \in H_A$,

(b_1) $F\ (h) < \infty$ implies $\hat{h} \leqslant h$ near ∞ and $F\ (\hat{h}) < \infty$

(b_2) $F\ (h) = \infty$ implies that $F\ \hat{h} = \infty$.

We omit the proof as it is obvious analogous of the proof of Lemma 2.14 of Jain and Taylor[7].

In this paper we shall denote various positive constants by the same symbol C.

3. Main Results

Theornm 3.1—Let

(i) (A_1) and (A_2) hold, (ii) $X(t)$ be a solution of (1.1) with $EX_0^2 < \infty$, (iii) $a(x, t) \geqslant 0$ for all t and x.

Then

$$| X(t) - W(t) | = o\left(t^{1/2} (\log\log t)^{-1/2}\right)$$

almost surely as $t \to \infty$.

Proof : We have

$$\frac{X(t) - W(t)}{t^{1/2} (\log\log t)^{-1/2}} = \frac{X_0}{t^{1/2} (\log\log t)^{-1/2}} + \frac{\int_0^t a\, x(s), s)\, ds}{t^{1/2} (\log\log) t)^{-1/2}}.$$

So for $t_m = m^\lambda$, $\lambda = 4/\delta$, m a positive integer,

$$P\{\sup_{t_m \leqslant t \leqslant t_{m+1}} \left| \frac{X(t) - W(t)}{t^{1/2} (\log\log t)^{-1/2}} \right| > \frac{1}{m}\}$$

$$\leqslant P\{\sup_{t_m \leqslant t \leqslant t_{m+1}} \left| \frac{X_0}{t^{1/2} (\log\log t)^{-1/2}} \right| > \frac{1}{2m}\}$$

$$+ P\{\sup_{t_m \leqslant t \leqslant t_{m+1}} \left| \frac{\int_0^t a(X(s), s)\, ds}{t^{1/2} (\log\log t_{m+1})^{-1/2}} \right| > \frac{1}{2m}\}$$

$$\leqslant P\left\{\frac{| X_0 |}{t_m^{1/2} (\log\log t)_{m+1}^{-1/2}} > \frac{1}{2m}\right\}$$

$$+ P\left\{\frac{\left| \int_0^{t_{m+1}} a(X(s(s), s\, ds \right|}{t_m^{1/2} (\log\log t_{m+1})^{-1/2}} > \frac{1}{2m}\right\}$$

$$\leqslant \frac{4m^2 (\log\log t_{m+1})}{t_m} EX_0^2$$

$$+ \frac{4m^2 (\log\log t_{m+1})}{t_m} E \left| \int_0^{t_{m+1}} a(X(s), s)\, ds \right|^2$$

(*equation continued on p.* 1039)

$$\leqslant \frac{Cm^2 (\log \log (m+1)^{4/\delta})}{m^{4/\delta}} + \frac{Cm^2 (\log \log (m+1)^{4/\delta})}{m^{4/\delta}} \times \left(t_{m+1}^{1-\delta} \right)$$

(by Lemma 1)

$$\leqslant \frac{C \log m)}{m^2}.$$

Now since $\sum_{m=1}^{\infty} C (\log m)/m^2 < \infty$, we have by applying Borel-Cantelli Lemma,

$$P \{ \sup_{t_m \leqslant t \leqslant t_{m+1}} \left| \frac{X(t) - W(t)}{t^{1/2} (\log \log t)^{-1/2}} \right| > \frac{1}{m} \text{i. o.}\} = 0,$$

and consequently

$$P \{ \lim_{t \to \infty} \left| \frac{X(t) - W(t)}{t^{1/2} \log \log t)^{-1/2}} \right| = 0\} = 1.$$

Theorem 3.2—Let

(i) (A_1) and (A_2) hold, (ii) $X(t)$ be a solution of (1.1) with $EX_0^2 < \infty$, (iii) $a(x, t) \geqslant 0$ for all t and x, (iv) $h(t) > 0$ increase monotonically to infinity with t.

Then

$$P \{X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty\} = 0 \text{ or } 1 \text{ according as}$$

$$I(h) = \int_1^{\infty} \frac{h(t)}{t} \exp \{- \frac{h^2(t)}{2}\} dt < \infty \text{ or } = \infty. \qquad \ldots(3.1)$$

PROOF : Let us assume that

$$h_1(t) \leqslant h(t) \leqslant h_2(t) \text{ for all } t \text{ sufficiently large} \qquad \ldots(3.2)$$

where $h_1(t) = (\log \log t)^{1/2}$ and $h_2(t) = 2 (\log \log t)^{1/2}$.

Let us first establish the theorem under assumption (3.2). We will then show that the theorem is true for any arbitrary increasing nonnegative function $h(t)$.

By Theorem 3.1, for any $\beta > 0$

$$| X(t) - W(t) | < \beta t^{1/2} (\log \log t)^{-1/2} \text{ a. s. as } t \to \infty.$$

i. e.

$$W(t) - \beta t^{1/2} (\log \log t)^{-1/2} < X(t) < W(t) + \beta t^{1/2} (\log \log t)^{-1/2}$$

$$\text{almost surely as } t \to \infty. \qquad \ldots(3.3)$$

Let us first consider the case when

$$X(t) < W(t) + \beta t^{1/2} (\log \log t)^{-1/2}$$

$$< W(t) + 2\beta t^{1/2} h^{-1}(t) \text{ (by relation (3.2))}.$$

Therefore

$$P\{X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty\}$$

$$\leqslant P\{W(t) > t^{1/2} (h(t) - \frac{2\beta}{h(t)}) \text{ i. o. as } t \to \infty\}. \qquad \ldots(3.4)$$

Since for $h(t)$ increasing, $h(t) - \dfrac{2\beta}{h(t)}$ is also increasing,

$$I(h) < \infty \Rightarrow I(h - \frac{2\beta}{h}) < \infty.$$

So by Kolmogorov's test criterion for Brownian motion, if $I(h) < \infty$, then

$$P\{W(t) > t^{1/2} (h(t) - \frac{2\beta}{h(t)}) \text{ i. o.as } t \to \infty\} = 0.$$

Therefore,

$$P\{X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty\} = 0.$$

Next let us consider the case when

$$W(t) - \beta t^{1/2} (\log \log t)^{-1/2} < X(t).$$

Since $a \geqslant 0$, by Theorem 3.1 the above expression can be written as

$$X(t) < W(t) + \beta t^{1/2} (\log \log t)^{-1/2}.$$

So when $I(h) < \infty$, we have

$$P\{X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty\} = 0$$

as shown above.

The fact that $I(h) = \infty \Rightarrow$

$$P\{X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty\} = 1$$

is trivial in view of Kolmogorov's test criterion for Brownian motion and the assumption that $a(x, t) \geqslant 0$ for all t and x.

Now let us remove the restriction (3.2) and consider $h(t)$ an arbitrary increasing nonnegative function.

Define,

$$\hat{h}(t) = \min [\max (h(t), h_1(t)), h_2(t))]. \qquad \ldots(3.5)$$

Then by Lemma 3,

$$I(h) < \infty \Rightarrow I(\hat{h}) < \infty \text{ and } \hat{h} \leqslant h \text{ near infinity.}$$

Again we have $h_1(t) \leqslant \hat{h}(t) \leqslant h_2(t)$.

Therefore,

$$P\{X(t) > t^{1/2}\,\hat{h}(t) \text{ i. o. as } t \to \infty\} = 0$$
$$\text{when } I(h) < \infty.$$

But $\hat{h}(t) \leqslant h(t)$ near infinity.

Hence,

$$P\{X(t) > t^{1/2}\,h(t) \text{ i. o. as } t \to \infty\} = 0 \qquad \ldots(3.6)$$

when $I(h) < \infty$.

Next again by Lemma 3,

$$I(h) = \infty \Rightarrow I(\hat{h}) = \infty.$$

So

$$P\{X(t) > t^{1/2}\,\hat{h}(t) \text{ i. o. as as } t \to \infty\} = 1.$$

This implies that there exists a sequence $\{t_n\} \uparrow \infty$ such that

$$X(t_n) > t_n^{1/2}\,\hat{h}(t_n) \text{ a. s. for every positive integer } n. \qquad \ldots(3.7)$$

Since $I(h_2) < \infty$, we have

$$P\{X(t) > t^{1/2}\,h_2(t) \text{ i. o. as } t \to \infty\} = 0.$$

So for $\{t_n\} \uparrow \infty$,

$$X(t_n) \leqslant t_n^{1/2}\,h_2(t_n) \text{ a. s. for every positive integer } n. \qquad \ldots(3.8)$$

Now from (3.7) and (3.8), we get

$$\hat{h}(t_n) \leqslant h_2(t_n) \text{ for every positive integer } n \text{ and}$$

hence,

$$\hat{h}(t_n) = \max\,[h(t_n), h_1(t_n)], \text{ by (3.5)}$$

i. e. $\hat{h}(t_n) \geqslant h(t_n)$ for every positive integer n.

Therefore by (3.7),

$$X(t_n) > t_n^{1/2} \, h(t_n) \text{ a. s. for every positive integer } n.$$

Hence for $I(h) = \infty$,

$$P\{X(t) > t^{1/2} \, h(t) \text{ i. o. as } t \to \infty\} = 1. \qquad ...(3.9)$$

From (3.6) and (3.9), it is evident that we have removed the restriction (3.2). Hence without any loss of generality we can assume (3.2), (for the proof of this statement we have followed the technique adopted by Jain *et al.*[6]

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REFERENCES

1. K. L. Chung, *Trans. Am. Math. Soc.* 64 (1948), 205–33.
2. W. Feller, *Trans. Am. Math. Soc.* 54 (1943), 373–402.
3. A. Friedman, *Stochastic Differential Equations and Applications.* Vol. 1. Academic Press, New York, 1975.
4. I. I. Gikhman and A. V. Skorohod, *Stochastic Differential Equations.* Springer Verlag, Berlin, 1972.
5. K. Ito and H. P. Mckean (Jr), *Diffusion Processes and Their Sample Paths.* Springer Verlag, Berlin, 1965.
6. N. C. Jain, K. Jogdeo and W. F. Stout, *Ann. Prob.* 3 (1974), 119–45.
7. N. C. Jain and S. J. Taylor, *Ann. Prob.* 1 (1973), 527–49.
8. M. N. Mishra and S. K. Acharya, *Indian J. pure. appl. Math.* 14 (1983), 1335–42.
9. V. Strassen, *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* 2 (1965), 315–43.

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ORDER LEVEL INVENTORY SYSTEM WITH POWER DEMAND PATTERN FOR ITEMS WITH VARIABLE RATE OF DETERIORATION

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The present paper deals with a power demand pattern inventory model with variable rate of deterioration. Both deterministic and probabilistic demands have been considered. Ultimately, some particular cases regarding the demand pattern have also been discussed.

1. Introduction

The effect of deterioration is very important in many inventory systems. Deterioration is defined as decay or damage such that the item can not be used for its original purpose. Food items, drugs, pharmaceuticals and radioactive substances are examples of items in which sufficient deterioration can take place during the normal storage period of the units and consequently this loss must be taken into account when analyzing the system. Efforts in analyzing mathematical models of inventory in which a constant or variable proportion of the on-hand inventory deteriorates with time have been undertaken by Ghare and Schrader[1], Goel and Aggarwal[2], Covert and Philip[3], Shah[4], Misra[5] etc. to name only a few. Covert and Philip in their paper have developed an economic order quantity model for items with variable rate of deterioration by assuming Weibull density function for the time of deterioration and a constant demand rate.

In the present paper attempts have been made to investigate an EOQ model assuming the existance of a suitable power demand pattern and a special form of Weibull density function. Such a special form is chosen in order to make the problem mathematically tractable. Deterministic as well as probabilistic cases of demands are considered allowing shortages. Ultimately, some particular cases of the power demand pattern have been discussed.

2. The Mathematical Model

The model is developed under the following assumptions :

(i) Replenishment size is constant and the replenishment rate is infinite;

(ii) lead time is zero;

(iii) T is the fixed length of each production cycle;

(iv) C_1 is the inventory holding cost per unit per unit time;

(v) C_2 is the shortage cost per unit per unit time;

(vi) C_3 is the cost of each deteriorated unit;

(vii) shortages are allowed and fully backlogged;

(viii) a variable fraction $\theta(t)$ of the on-hand inventory deteriorates per unit time.

In the present model, the function $\theta(t)$ is assumed in the form

$$\theta(t) = \theta_0 t;\ 0 < \theta_0 \leqslant 1,\ t > 0$$

which is a special form of two parameter Weibull function considered by Covert and Philip[3];

(ix) the demand upto time t is assumed to be $d\left(\frac{t}{T}\right)^{1/n}$, vide, Naddor[6] where d is the demand size during the fixed cycle time T and n $(0 < n < \infty)$ is the pattern index. $(dt^{(1-n)/n})/nT^{1/n}$ is the demand rate at time t. Such pattern in the demand rate is called power demand pattern.

3. Deterministic Demand

Let Q be the total amount of inventory produced or purchased at the beginning of each period and after fulfilling backorders let us assume we get an amount S $(S > 0)$ as initial inventory. Let d be the demand during period T. Inventory level gradually diminishes during time period $(0, t_1)$, $t_1 < T$ due to the reasons of market demand and deterioration of the items and ultimately falls to zero at time $t = t_1$. Shortages occur during time period (t_1, T) which are fully backlogged. Let $I(t)$ be the on-hand inventory at any time t. The differential equations which the on-hand inventory $I(t)$ must satisfy in two different parts of the cycle time T are the following:

$$\frac{dI(t)}{dt} + \theta(t)\, I(t) = -\frac{dt^{(1-n)/n}}{nT^{1/n}},\ 0 \leqslant t \leqslant t_1 \qquad \text{...(3.1)}$$

and

$$\frac{dI(t)}{dt} = -\frac{dt^{(1-n)/n}}{nT^{1/n}},\ t_1 \leqslant t \leqslant T. \qquad \text{...(3.2)}$$

Solutions of the above differential equations are

$$I(t) = S\exp\left(-\frac{\theta_0}{2}t^2\right) - \frac{d\exp\left(-\frac{\theta_0}{2}t^2\right)}{nT^{1/n}}\int_0^t t^{(1-n)/n}$$

$$\times \exp\left(\frac{\theta_0}{2}t^2\right) dt,\ 0 \leqslant t \leqslant t_1 \qquad \text{...(3.3)}$$

and

$$I(t) = \frac{d}{T^{1/n}}\left(t_1^{1/n} - t^{1/n}\right), \; t_1 \leqslant t \leqslant T. \qquad \ldots(3.4)$$

Since $I(t_1) = 0$, we find neglecting higher order terms of θ_0 ($\ll 1$) the following :

$$S = \frac{dt_1^{1/n}}{T^{1/n}} + \frac{\theta_0 d}{2(2n+1)T^{1/n}} t_1^{(1+2n)/n}. \qquad \ldots(3.5)$$

Hence the total amount of deteriorated units

$$= S - \int_0^{t_1} \frac{dt^{(1-n)/n}}{nT^{1/n}} \, dt = S - \frac{dt_1^{1/n}}{T^{1/n}} = \frac{\theta_0 \, dt_1^{(1+2n)/n}}{2(2n+1)T^{1/n}}. \qquad \ldots(3.6)$$

Average total cost per unit time is given by

$$C(S, t_1) = \frac{C_3 \, \theta_0 \, dt_1^{(1+2n)/n}}{2T^{(1+n)/n}(2n+1)} + \frac{C_1}{T}\int_0^{t_1} I(t)\, dt - \frac{C_2}{T}\int_{t_1}^{T} I(t)\, dt.$$

Now substituting the values for $I(t)$ given by eqns (3.3), (3.4), eliminating S using eqn. (3.5) and then on integration we find

$$\begin{aligned} C(t_1) = {} & \frac{C_3\,\theta_0 d}{2(2n+1)T^{(1+n)/n}} t_1^{(1+2n)/n} + \frac{C_1 d}{T^{(1+n)/n}} t_1^{(1+n)/n} \\ & + \frac{C_1\,\theta_0\, d}{2(2n+1)T^{(1+n)/n}} t_1^{(1+3n)/n} - \frac{C_1\,\theta_0\, d}{6T^{(1+n)/n}} t_1^{(1+3n)/n} \\ & - \frac{C_1\, nd}{(n+1)T^{(1+n)/n}} t_1^{(1+n)/n} + \frac{C_1\, n^2\,\theta_0\, d}{(2n+1)(3n+1)T^{(1+n)/n}} t_1^{(1+3n)/n} \\ & + \frac{C_2\, nd}{n+1} - \frac{C_2\, d}{T^{1/n}} t_1^{1/n} + \frac{C_2\, d}{(n+1)T^{(1+n)/n}} t_1^{(1+n)/n}. \end{aligned}$$

[neglecting terms containing higher powers of θ_0]. ...(3.7)

For minimum, the necessary condition is

$$\frac{dC(t_1)}{dt_1} = 0.$$

This gives

$$\frac{t_1^{(1-n)/n}}{T^{(1+n)/n}}\left[\frac{C_3\,\theta_0\, d}{2n} t_1^2 + \frac{C_1(n+1)d}{n} t_1 + \frac{C_1\,\theta_0\,(3n+1)\, d}{2n(2n+1)} t_1^3\right.$$

(*equation continued on p.* 1046)

$$- C_1 dt_1 + \frac{C_1 n \theta_0 d}{2n+1} t_1^3 - \frac{C_1 \theta_0 (3n+1) d}{6n} t_1^3 - \frac{C_2 dT}{n}$$

$$+ \frac{C_2 d}{n} t_1 \Big] = 0.$$

But as $t_1 > 0$ we find from the above equation the following cubic in t_1.

$$Lt_1^3 + M t_1^2 + Nt_1 + P = 0 \qquad \text{...(3.8)}$$

where

$$\left.\begin{aligned} L &= \frac{C_1 \theta_0 d}{3n}, \\ M &= \frac{C_3 \theta_0 d}{2n}, \\ N &= \frac{d}{n}(C_1 + C_2), \\ P &= -\frac{C_2 dT}{n} \end{aligned}\right\}. \qquad \text{...(3.9)}$$

Since (3.8) is a cubic equation in t_1 with last term P negative, it has at least one positive root. Again since $0 < \theta_0 \leqslant 1$, we find $LN - M^2 > 0$ and so other two roots of (3.8) are imaginary. Let t_{10} be the positive root of (3.8).

$\therefore$ Optimum value of t_1 is $t_1^* = t_{10}$. Substituting it in (3.5), the optimum S is

$$S^* = \frac{dt_1^{*1/n}}{T^{1/n}} + \frac{\theta_0 d}{2(2n+1) T^{1/n}} t_1^{*(1+2n)/n} \qquad \text{...(3.10)}$$

The relation connecting S and Q is the following

$$Q = S + d - \frac{dt_1^{1/n}}{T^{1/n}}. \qquad \text{...(3.11)}$$

Hence the optimum value for Q is

$$Q^* = d + \frac{\theta_0 d}{2(2n+1)T^{1/n}} t_1^{*(1+2n)/n}. \qquad \text{...(3.12)}$$

Minimum value of C is $C\left(t_1^*\right)$.

If there be no deterioration, then $\theta_0 = 0$.

$\therefore$ From (3.8) [using (3.9)]

$$t_1^* = \frac{C_2 T}{C_1 + C_2}$$

and corresponding expressions for Q^* and S^* can be obtained by substituting $\theta_0 = 0$ in the expressions (3.12) and (3.10) respectively.

4. Probabilistic Demand

In this case, it is assumed that the demand during the period $(0, T)$ is a random variable x with probability density function $f(x)$ $(0 < x < \infty)$ and the demand follows power demand pattern with the demand rate $(xt^{(1-n)/n})/nT^{1/n}$. Solution of the problem has been investigated under the following two cases.

Case 1 : When no shortages occur

Let us assume the inventory level of the system at any time t $(0 \leqslant t \leqslant T)$ to be $I_{1x}(t)$. Hence the differential equation which would govern the system would be

$$\frac{dI_{1x}(t)}{dt} + \theta_0\, t I_{1x}(t) = -\frac{xt^{(1-n)/n}}{nT^{1/n}},\ 0 \leqslant t \leqslant T. \qquad \text{...(4.1)}$$

Solution of the equation (4.1) is the following :

$$I_{1x}(t) = S \exp\left(-\frac{\theta_0}{2}t^2\right) - \frac{x \exp\left(-\frac{\theta_0}{2}t^2\right)}{nT^{1/n}} \int_0^t t^{(1-n)/n} \exp\left(\frac{\theta_0}{2}t^2\right) dt,$$

$$0 \leqslant t \leqslant T \qquad \text{...(4.2)}$$

where S (> 0) is the expected stock on hand at the beginning after meeting backorders. Since there is no shortage, we have

$$I_{1x}(T) \geqslant 0$$

or,

$$S - \frac{x}{n\,T^{1/n}} \int_0^T t^{(1-n)/n} \exp\left(\frac{\theta_0}{2}t^2\right) dt \geqslant 0$$

or,

$$x \leqslant S_1,$$

where

$$S_1 = \frac{SnT^{n/1}}{\int_0^T t^{(1-n)/n} \exp\left(\frac{\theta_0}{2}t^2\right) dt}.$$

The expression for S_1 can be simplified further by neglecting higher powers of θ_0. We ultimately obtain

$$S_1 = S\left[1 - \frac{\theta_0 T^2}{2(2n+1)}\right]. \qquad \ldots(4.3)$$

The average number of items $H_1(x)$ carried in inventory per unit time is the following

$$H_1(x) = \frac{1}{T}\int_0^T I_{1x}(t)\,dt,\ x \leqslant S_1$$

$$= S - \frac{S\theta_0 T^2}{6} - \frac{nx}{n+1} + \frac{x\,\theta_0\,n^2T^2}{(2n+1)(3n+1)},\ x \leqslant S_1. \qquad \ldots(4.4)$$

Average number of items that deteriorates per unit time is

$$D_1(x) = \frac{1}{T}[S - x - I_{1x}(T)]$$

$$= \tfrac{1}{2}S\theta_0 T - \frac{n\,\theta_0\,T}{2n+1}x,\ x \leqslant S_1. \qquad \ldots(4.5)$$

Average shortage per unit time is

$$G_1(x) = 0. \qquad \ldots(4.6)$$

Case 2 : When shortages occur

If the system carries inventory during the period $(0, t_1)$ and then shortages occur for the remaining period (t_1, T) of the cycle then the inventory level $I_{2x}(t)$ at any instant t would satisfy similar differential equations as in the deterministic case and their solutions would be

$$I_{2x}(t) = S\exp\left(-\frac{\theta_0}{2}t^2\right) - \frac{x\exp\left(-\frac{\theta_0}{2}t^2\right)}{nT^{1/n}}\int_0^t t^{(1-n)/n}\exp\left(\frac{\theta_0}{2}t^2\right)dt,$$

$$0 \leqslant t \leqslant t_1 \qquad \ldots(4.7)$$

and

$$I_{2x}(t) = \frac{x}{T^{1/n}}\left(t_1^{1/n} - t^{1/n}\right),\ t_1 \leqslant t \leqslant T. \qquad \ldots(4.8)$$

Since shortages occur, we must have

$$I_{2x}(T) < 0$$

or

$$x > S_1, \text{ where } S_1 \text{ is given in (4.3).}$$

Again, $I_{2x}(t_1) = 0$. This gives

$$S - \frac{x}{nT^{1/n}} \int_0^{t_1} t^{(1-n)/n} \exp\left(\frac{\theta_0}{2} t^2 \right) dt = 0.$$

Expanding the integrand in ascending powers of θ_0 and then integrating and neglecting all higher order terms in θ_0 we get

$$t_1^{1/n}\left[1 + \frac{\theta_0}{2(2n+1)} t_1^2\right] = \frac{S}{x} T^{1/n}. \quad \text{...(4.9)}$$

Taking nth power of both sides and then simplifying we find

$$t_1\left[1 + \frac{n\theta_0}{2(2n+1)} t_1^2\right] = \left(\frac{S}{x}\right)^n T$$

or

$$t_1^3 + 2Ht_1 + G = 0 \quad \text{...(4.10)}$$

where

$$H = \frac{2(2n+1)}{3\theta_0 n}, \; G = -\frac{2(2n+1)}{n\theta_0}\left(\frac{S}{x}\right)^n T.$$

Solving the above cubic equation (4.10) by Cardon's method we find the positive real root as follows :

$$t_1 = u - \frac{2(2n+1)}{3n\,\theta_0}\frac{1}{u}, \quad \text{...(4.11)}$$

where

$$u^3 = \tfrac{1}{2}\left[-G + \sqrt{(G^2 + 4H^3)}\right]$$

$$= \tfrac{1}{2}\left[\frac{2(2n+1)}{n\theta_0} . T\left(\frac{S}{x}\right)^n + \left\{\frac{4(2n+1)^2 T^2}{n^2\,\theta_0^2}\left(\frac{S}{x}\right)^{2n} + \frac{32}{27}\frac{(2n+1)^3}{n^3\,\theta_0^2}\right\}^{1/2}\right].$$

Expressing u and $\frac{1}{u}$ in a series in ascending powers of θ_0 and then substituting them in the equation(4.11) we find

$$t_1 = \left(\frac{S}{x}\right)^n T - \frac{21}{16}\left(\frac{n\theta_0}{2n+1}\right) T^3 \left(\frac{S}{x}\right)^{3n}, \quad \text{...(4.12)}$$

neglecting all higher order terms in θ_0. From the above equation the expressions for $t_1^{1/n}$, $t_1^{(1+n)/n}$, $t_1^{(1+3n)/n}$ etc. which are needed to calculate the total average expected cost can be obtained as follows :

$$t_1^{1/n} = \frac{S}{x} T^{1/n} - \frac{21}{16} \frac{\theta_0}{(2n+1)} T^{(1+2n)/n} \left(\frac{S}{x} \right)^{3n} \qquad \text{...(4.13)}$$

$$t_1^{(1+n)/n} = \left(\frac{S}{x} \right)^{n+1} T^{(1+n)/n} - \frac{21}{16} \frac{(n+1)\,\theta_0}{(2n+1)} T^{(1+3n)/n} \left(\frac{S}{x} \right)^{3n+1} \qquad \text{...(4.14)}$$

$$t_1^{(1+3n)/n} = \left(\frac{S}{x} \right)^{3n+1} T^{(1+3n)/n} - \frac{21}{16} \left(\frac{3n+1}{2n+1} \right) \theta_0 \, T^{(1+5n)/n} \left(\frac{S}{x} \right)^{5n+1}. \qquad \text{...(4.15)}$$

The following expression for S in terms of t_1 can be obtained from eqn. (4.9)

$$S = \left[\frac{t_1^{1/n}}{T^{1/n}} + \frac{\theta_0 \, t_1^{(1+2n)/n}}{2\,(2n+1) T^{1/n}} \right] . \, x. \qquad \text{...(4.16)}$$

The average number of items $H_2(x)$ carried in inventory per unit time is the following:

$$H_2(x) = \frac{1}{T} \int_0^{t_1} I_{2x}(t)\, dt, \; x > S_1$$

$$= \frac{x}{T^{(1+n)/n}} \left[\frac{1}{n+1} t_1^{(1+n)/n} + \frac{\theta_0}{3\,(3n+1)} t_1^{(1+3n)/n} \right], \; x > S_1. \qquad \text{...(4.17)}$$

Average amount of inventory that deteriorates per unit time is

$$D_2(x) = \frac{1}{T} \left[S - \int_0^{t_1} \frac{x t^{(1-n)/n}}{n T^{1/n}} \, dt \right], \; x > S_1$$

$$= \frac{S}{T} - \frac{x t_1^{1/n}}{T^{(1+n)/n}}. \qquad \text{...(4.18)}$$

Average shortages per unit time is

$$G_2(x) = = -\frac{1}{T} \int_{t_1}^{T} I_{2x}(t)\, dt, \; x > S_1$$

$$= \frac{x}{T^{(1+n)/n}} \left[\frac{n}{n+1} T^{(1+n)/n} - T t_1^{1/n} + \frac{1}{n+1} t_1^{(1+n)/n} \right]. \quad x > S_1 \qquad \text{...(4.19)}$$

∴ Expected total cost of the system per unit time becomes [using (4.3) for S_1]

$$K(t_1, S) = C_3 \int_0^{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]} D_1(x) f(x)\, dx + C_3 \times \int_{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]}^{\infty} D_2(x) f(x)\, dx + C_1 \times \int_0^{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]} H_1(x) f(x)\, dx$$

$$+ C_1 \int_{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]}^{\infty} H_2(x) f(x)\, dx + C_2 \int_0^{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]} G_1(x) f(x)\, dx + C_2 \int_{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]}^{\infty} G_2(x) f(x)\, dx.$$

Now substituting the values of $D_1(x)$, $D_2(x)$, $H_1(x)$, $H_2(x)$, $G_1(x)$, $G_2(x)$ from (4.5), (4.18), (4.4), (4.17), (4.6), (4.19) respectively and finally eliminating t_1 using (4.12), (4.13) (4.14) (4.15) we get the following :

$$K(S) = C_3 \int_0^{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]} \left[\tfrac{1}{2} S\theta_0 T - \frac{n\theta_0 T}{2n+1} x\right] f(x)\, dx$$

$$+ C_3 \int_{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]}^{\infty} \left[\frac{21}{16} \frac{\theta_0}{2n+1} T \frac{S^{3n}}{x^{3n-1}}\right] \times f(x)\, dx$$

$$+ C_1 \int_0^{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]} \left[S - \frac{S\theta_0 T^2}{6} - \frac{nx}{n+1} + \frac{\theta_0 n^2 T^2}{(2n+1)(3n+1)} x\right] f(x)\, dx$$

(equation continued on p. **1052)**

$$+ C_1 \int_{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]}^{\infty} \left[\frac{1}{n+1}\frac{S^{n+1}}{x^n} - \frac{(157n+47)\,\theta_0}{48\,(2n+1)\,(3n+1)} T^2 \frac{S^{3n+1}}{x^{3n}}\right] f(x)\, dx$$

$$+ C_2 \int_{S\left[1-\frac{\theta_0 T^2}{2(2n+1)}\right]}^{\infty} \left[\frac{nx}{n+1} - S + \frac{21\theta_0 T^2}{16\,(2n+1)} \frac{S^{3n}}{x^{3n-1}} + \frac{1}{n+1}\frac{S^{n+1}}{x^n} - \frac{21\theta_0 T^2}{16\,(2n+1)}\frac{S^{3n+1}}{x^{3n}}\right] f(x)\, dx. \quad \ldots(4.20)$$

If the probability density function $f(x)$ and pattern index n are prescribed, then right-hand side of eqn. (4.20) can be evaluated. The necessory condition for the minimum expected cost $K(S)$ is the relation

$$\frac{dK(S)}{dS} = 0.$$

Equating $\frac{dK(S)}{dS}$ to zero, the optimum value of $S = S^*$ (> 0) can be derived. For this value of $S = S^*$, the sufficient condition for minimum $\left.\frac{d^2 K(S)}{ds^2}\right|_{S=S^*} > 0$ would also be satisfied.

5. Discussion

In the present problem, a power demand pattern has been assumed with demand rate $(dt^{(1-n)/n})/nT^{1/n}$, where T, d, t are prescribed cycle time, entire demand during $(0, T)$ period, t is time ($0 \leqslant t \leqslant T$) respectively and n is pattern index. Substituting different values to the pattern index n in the equations for the total cost and total expected cost per unit time given by the equations (3.7) and (4.20) respectively, we can determine the corresponding cost equations. Then differentiating and proceeding in the usual manner the optimum values S^*, t_1^* Q^* etc., can be evaluated. Substituting $n = 0$ and $n = \infty$ in the power demand pattern formula it can be seen that these two correspond to the two extreme cases, i. e., when the entire demand occurs at the end of the period and when the demand is instantaneous in nature. For $n = 1$ the demand pattern is uniform and if there is no deterioration then it corresponds to the case discussed in Wilson's model. In this case the total cost per unit time (deterministic case) given by (3.7) reduces to [by substituting $n = 1$, $\theta_0 = 0$]

$$C(t_1) = \frac{C_3 d}{T} + \frac{C_2 d}{2} + (C_1 + C_2)\frac{d}{2T^2} t_1^2 - \frac{C_2 d}{T} t_1.$$

For optimum C, $\dfrac{dC}{dt_1} = 0.$

Equating the derivative to zero and simplifying we find the optimum t_1 as

$$t_1 = t_1^* = \frac{C_2 T}{C_1 + C_2}.$$

Since $\left[\dfrac{d^2 C}{dt_1^2}\right]_{t_1 = t_1^*} > 0$, C would be minimized for $t_1 = t_1^*$.

For $n = 1$, $\theta_0 = 0$ we find from equation (3.10)

$$S^* = \frac{dt_1^*}{T} = \frac{d}{T}\,\frac{C_2 T}{C_1 + C_2} = \frac{C_2 d}{C_1 + C_2}.$$

Similarly, the expected total cost per unit time (probabilistic case) given by (4.20) reduces to

$$K(S) = C_1 \int_0^S \left[S - \frac{x}{2}\right] f(x)\,dx + C_1 \int_S^\infty \frac{S^2}{2x} f(x)\,dx$$

$$+ C_2 \int_S^\infty \left[\frac{x}{2} - S + \frac{S^2}{2x}\right] f(x)\,dx.$$

Solving $\dfrac{dK(S)}{dS} = 0$ we can determine the value of S^*.

When $n > 1$, a larger portion of the demand occurs towards the beginning of the period and when $0 < n < 1$, a larger portion of the demand occurs at the end of the period.

References

1. P. M. Ghare and G. F. Schrader, *J. Ind. Engng.* 14 (1983), 238-43.
2. V. P. Goel and S. P. Aggarwal, *Proceedings All India Seminar on Operational Research and Decision Making*, March 1981.
3. R. P. Covert and G. C. Philip, *AIIE Trans*, 5 (1973), 323-26.
4. Y K. Shah, *AIIE Trans.* 9 (1977) 108-12.
5. R. B. Misra, *Int. J. Prod. Res.* 13 (1975), 495-505.
6. E. Naddor, *Inventory Systems*. John Wiley and Sons, New York, 1966.

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ON THE EQUICONVERGENCE OF THE EIGENFUNCTION EXPANSION ASSOCIATED WITH CERTAIN 2ND ORDER DIFFERENTIAL EQUATIONS

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We consider the differential equation

$$L[y] = -\frac{d^2 y(x)}{dx^2} + q(x)\, y(x) = \lambda\, y(x),\ 0 \leqslant x < \infty$$

where λ is a complex parameter. It has been shown that the convergence of the expansion of any function $f(.)$, square integrable on $[0, \infty)$, in terms of the eigenfunctions of a boundary value problem associated with the differential equation is independent of the function $q(.)$, provided $q(.)$ is square integrable on $[0, \infty)$.

§ 1. We consider the differential equation

$$L[y] = -\frac{d^2 y(x)}{dx^2} + q(x)\, y(x) = \lambda\, y(x),\ 0 \leqslant x < \infty \qquad \ldots(1.1)$$

where $\lambda = u + iv$ is a complex parameter. One of the usual problems with such a differential equation is to consider the convergence of the expansion of an arbitary function $f(.)$, square-integrable on $[0, \infty)$, [we write $f \in L^2\,[0, \infty)$], in terms of the eigenfunctions of boundary value problem associated with the differential equation (1.1). The idea here is to prove the following :

Theorem—If $q(.) \in L^2\,[0, \infty)$, and the eigenfunction expansion of an arbitrary $f \in L^2\,[0, \infty)$ associated with a boundary value problem (II) consisting of the differential equation

$$L_0[y] = -\frac{d^2 y(x)}{dx^2} = \lambda\, y(x),\ 0 \leqslant x < \infty \qquad \ldots(1.2)$$

and the boundary condition

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0 \ (0 \leqslant \alpha < \pi) \qquad \ldots(1.3)$$

is convergent, then so is the eigenfunction expansion of f associated with the boundary value problem (I) consisting of (1.1) and the same boundary condition (1.3) as in the boundary value problem (II).

To establish this result, we obtain a relation between the Φ-functions (for the definition of the Φ-function we refer to section (2.6) of Titchmarsh[1] for these two boundary value problems and show that the contribution corresponding to the difference of these two Φ-functions towards the expansion of f is nil.

§ 2. Let $\phi \equiv \phi(x, \lambda)$ and $\theta \equiv \theta(x, \lambda)$ be the two linearly independent solutions of the differential equation (1.1) satisfying the following initial conditions :

$$\left.\begin{array}{l} \phi(0, \lambda) = \sin\alpha, \ \theta(0, \lambda) = \cos\alpha \\ \phi^{(1)}(0, \lambda) = -\cos\alpha, \ \theta^{(1)}(0, \lambda) = \sin\alpha \end{array}\right\} -\pi < \alpha < \pi. \qquad \ldots(2.1)$$

It has been shown by Titchmash[1] (§ 2.1) that there exists a function $m(.)$, analytic in the two half planes im $\lambda > 0$ and im $\lambda < 0$ such that, for im $\lambda \neq 0$

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\,\phi(x, \lambda) \in L^2[0, \infty). \qquad \ldots(2.2)$$

With the help of these two solutions $\phi(x, \lambda)$ and $\psi(x, \lambda)$ for any $f \in L^2[0, \infty)$ let us define the function $\Phi(x, \lambda; f) = \Phi(x)$ as

$$\Phi(x) = \psi(x, \lambda)\int_0^x \phi(y, \lambda) f(y)\, dy + \phi(x, \lambda)\int_x^\infty \psi(y, \lambda) f(y)\, dy.$$

This is the so called Φ-function of Titchmarsh associated with the BVP (I). It can be easily proved that $\Phi(x)$ satisfies the following differential equation

$$L[y] = \lambda y(x) - f(x). \qquad \ldots(2.3)$$

Titchmarsh[1] (§ 3.1) has proved that the expansion of the arbitrary function $f \in L^2[0, \infty)$ associated with the boundary value problem (I) is obtained from

$$\frac{i}{\pi}\int_{-R+i\delta}^{R+i\delta} \Phi(x, \lambda; f)\, d\lambda \qquad \ldots(2.4)$$

by making $R \to \infty$ and $\delta \to 0$.

Let $\phi_0(x, \lambda)$, $\theta_0(x, \lambda)$ be the solutions of the differential equation (1.2) satisfying the following :

$$\begin{array}{l} \phi_0(0, \lambda) = \sin\alpha, \ \phi_0^{(1)}(0, \lambda) = -\cos\alpha \\ \theta_0(0, \lambda) = \cos\alpha, \ \theta_0^{(1)}(0, \lambda) = \sin\alpha \end{array} \quad -\pi < \alpha < \pi.$$

It follows easily that

$$\theta_0(x, \lambda) = \cos\alpha \cos(x\sqrt{\lambda}) + \frac{1}{\sqrt{\lambda}}\sin\alpha \sin(x, \sqrt{\lambda})$$

$$\phi_0(x, \lambda) = \sin\alpha \cos(x\sqrt{\lambda}) - \frac{1}{\sqrt{\lambda}}\cos\alpha \sin(x\sqrt{\lambda}). \qquad \ldots(2.5)$$

So, the square integrable solution $\psi_0(x, \lambda)$ of this differential equation is given by (Titchmarsh[1], § 4.1)

$$\psi_0(x, \lambda) = \frac{c^{ix\sqrt{\lambda}}}{\cos\alpha + i\sqrt{\lambda}\sin\alpha}. \qquad ...(2.6)$$

As $q \in L^2[0, \infty)$, we know that the differential equation (1.1) is separated i.e. f and $L[f] \in L^2[0, \infty)$ should imply that $qf \in L^2[0, \infty)$.

Since for each $f \in L^2[0, \infty)$ we know that $\Phi(x, \lambda; f) \in L^2[0, \infty)$ it is follows that $\Phi(x, \lambda; f)$ and $L[\Phi(x, \lambda; f)] = \lambda\Phi(x, \lambda; f) - f \in L^2[0, \infty)$ and by seperability of the differential equation (1.1) we have $q\Phi(x, \lambda; f) \in L^2[0, \infty)$. Now

$$\Phi(x, \lambda, q\Phi(x)) = \psi_0(x, \lambda)\int_0^x \phi_0(y, \lambda)\, q(y)\, \Phi(y)\, dy$$

$$+ \phi_0(x, \lambda)\int_0^\infty \psi_0(y, \lambda)\, q(\lambda)\, \Phi(y)\, dy. \qquad ...(2.7)$$

Since Φ is a solution of the differential equation (2.3), we get

$$L[\Phi(x)] = \lambda\Phi(x) - f(x)$$

or

$$L_0[\Phi(x)] + q(x)\Phi(x) = \lambda\Phi(x) - f(x)$$

or

$$L_0[\Phi(x)] = \lambda\Phi(x) - g(x)$$

where $g(x) = f(x) + q(x)\Phi(x) \in L^2[0, \infty)$. This implies that $\Phi(x)$ is the Φ-function of the boundary value problem associated with the differential equation (1.2) corresponding to g (Note that both the boundary value problems have the same boundary conditions). So,

$$\Phi(x, \lambda; f) = \Phi_0(x, \lambda; g) = \Phi_0(x, \lambda; f + q\Phi)$$

$$= \Phi_0(x, \lambda; f) + \Phi_0(x, \lambda; q\Phi). \qquad ...(2.8)$$

In view of (2.4) and (2.8), it now follows clearly that in order to establish the theorem we need only to show that the contribution towards eigenfunction expansion arising from $\Phi_0(x, \lambda; q\Phi)$ is nil.

The following two lemmas will be useful for further discussion.

Lemma 1—For any $f \in L^2[0, \infty)$, $\lambda = u + iv$,

$$\int_0^\infty |\Phi(x, \lambda; f)|^2\, dx \leqslant \frac{1}{v^2}\int_0^\infty |f(x)|^2\, dx.$$

For proof we refer to § 2.8 of Titchmarsh[1].

Lemma 2—If $\sqrt{\lambda} = s = \sigma + it$, $\lambda = u + iv$ then

$$\Phi_0 (x, \lambda; q\, \Phi) = O\left(\frac{1}{|\, v\,|\ |\, s\,|} \right).$$

PROOF:

$$\Phi_0 (x, \lambda; q\, \Phi) = \frac{e^{ixs}}{\cos \alpha + is \sin \alpha} \int_0^x \left\{ \sin \alpha \cos (ys) - \frac{1}{s} \cos \alpha \sin (ys) \right\} q\,(y)\, \Phi\,(y)\, dy$$

$$+ \frac{\left\{ \sin \alpha \cos (xs) - \frac{1}{s} \cos \alpha \sin (xy) \right\}}{\cos \alpha + \text{is} \sin \alpha} \int_x^\infty e^{iys}.$$

Now $\sin \alpha \cos ys - \frac{1}{s} \cos \alpha \sin ys$

$$= \frac{\sin \alpha}{2} (e^{iys} + e^{-iys}) - \frac{\cos \alpha}{2is} (e^{iys} - e^{-iys})$$

$$= \left(\frac{\sin \alpha}{2} - \frac{\cos \alpha}{2is} \right) e^{iys} + \left(\frac{\sin \alpha}{2} + \frac{\cos \alpha}{2is} \right) e^{-iys}$$

$$= O\left(e^{-ty} \left| \frac{\sin \alpha}{2} - \frac{\cos \alpha}{2is} \right| \right) + O\left(e^{ty} \left| \frac{\sin \alpha}{2} + \frac{\cos \alpha}{2is} \right| \right)$$

$$= O\,(e^{ty}).$$

So,

$$e^{ixs} \left\{ \sin \alpha \cos ys - \frac{\cos \alpha}{s} \sin ys \right\}$$

$$= O\,(e^{-tx})\, O\,(e^{ty}) = O\,(1) \text{ since } 0 \leqslant y \leqslant x.$$

Similarly

$$\sin \alpha \cos (xs) - \frac{\cos \alpha}{s} \sin (xs) = O\,(e^{tx})$$

and so

$$\left\{ \sin \alpha \cos (xs) - \frac{\cos \alpha}{s} \sin (xs) \right\} e^{iys} = O\,(e^{tx}\, e^{-tx}) \text{ for all } y \geqslant x$$

$$= O\,(1).$$

Hence

$$\Phi_0 (x, \lambda; q\, \Phi) = O\left(\frac{1}{(\cos \alpha + is \sin \alpha)} \int_0^\infty |\, q\,(y)\, \Phi\,(y)\,|\, dy) \right)$$

(*equation continued on p.* 1058)

$$= O\left(\frac{1}{|s|}\left\{\int_0^\infty |q(y)|^2\,dy\right\}^{1/2}\left\{\int_0^\infty |\Phi(y,\lambda;f)|^2\,dy\right\}^{1/2}\right)$$

$$= O\left(\frac{1}{|v|\ |s|}\right), \text{ using Lemma 1, for all } \lambda. \qquad \ldots(2.9)$$

§ 3. Now consider the contour $\Gamma(R)$ formed by the segments of lines $(-R+i, -R+i\delta)$, $(-R+i\delta, R+i\delta)$, $(R+i\delta, R+i)$ joined by semicircles of radius R and centres $\pm i$. Using (2.8) we get,

$$\int_{\Gamma(R)} \Phi(y,\lambda;f)\,d\lambda = \int_{\Gamma(R)} \Phi_0(x,\lambda;f)\,d\lambda - \int_{\Gamma(R)} \Phi_0(x,\lambda;q\,\Phi)\,d\lambda. \qquad \ldots(3.1)$$

Using Lemma 2 we get

$$\int_{\Gamma(R)} \Phi_0(x,\lambda;q(x)\,\Phi(x))\,d\lambda = O\left(\int_{\Gamma(R)} \frac{1}{|\sqrt{\lambda}|\ |v|}\,d\lambda\right). \qquad \ldots(3.2)$$

On the part of the upper semicircle in the 1st quadrant, we have

$$\lambda = i + Re^{i\phi}\left(0 \leqslant \phi \leqslant \frac{\pi}{2}\right).$$

Hence, $\int_{\Gamma(R)} \Phi_0(x,\lambda;q(x)\,\Phi(x,\lambda;f))\,d\lambda$ integrated round this semicircle in the first quadrant, gives,

$$O\left\{\int_0^{\pi/2} \frac{R\,d\phi}{R^{1/2}(1+R\sin\phi)}\right\} = O\left\{\int_0^{\pi/2} \frac{R^{1/2}\,d\phi}{1+R\sin\phi}\right\} = o(1) \text{ as } R \to \infty.$$

Hence finally it is a question of proving that

$$\lim_{\substack{\delta\to 0\\ R\to\infty}} \int_{R+i\delta}^{R+i} \Phi_0(x,\lambda,q(x)\,\Phi(x,\lambda;f))\,d\lambda = 0. \qquad \ldots(3.3)$$

Using Lemma 2 we get

$$\int_{R+i\delta}^{R+i} \Phi_0(x,\lambda;q(x)\,\Phi(x))\,d\lambda = \int_{R+i\delta}^{R+i} O\left(\frac{1}{|\sqrt{\lambda}|\ |v|}\right)d\lambda$$

$$= \int_\delta^1 O\left(\frac{dv}{[R^2+v^2]^{1/4}\,|v|}\right), \text{ since } \lambda = u + vi$$

(*equation contiuned on p.* 1059)

$$= O\left(\int_{\delta}^{1} \frac{dv}{|v|\,|R^2+v^2|^{1/4}}\right)$$

$$= O\left(\int_{(R^2+\delta^2)^{1/4}}^{(R^2+1)^{1/4}} \frac{2t^2\,dt}{(t^2-R)\,(t^2+R)}\right),$$

substituting $R^2 + v^2 = t^4$

$$= O\left(\int_{(R^2+\delta^2)^{1/4}}^{(R^2+1)^{1/2}} \frac{(t^2-R)+(t^2+R)}{(t^2-R)\,(t^2+R)}\,dt\right)$$

$$= O\left(\frac{1}{\sqrt{R}}\left\{\tan^{-1}\frac{(R^2+1)^{1/4}}{\sqrt{R}} - \tan^{-1}\frac{(R^2+\delta^2)^{1/4}}{\sqrt{R}}\right\}\right)$$

$$+ O\left(\frac{1}{\sqrt{R}}\log\frac{\{(R^2+1)^{1/4}-\sqrt{R}\}\,\{(R^2+\delta^2)^{1/4}+\sqrt{R}\}}{\{(R^2+1)^{1/4}+\sqrt{R}\}\,\{(R^2+\delta^2)^{1/4}-\sqrt{R}\}}\right)$$

$$= O\left(\frac{1}{\sqrt{R}}\tan^{-1}\frac{\left(1+\frac{1}{R^2}\right)^{1/4}-\left(1+\frac{\delta^2}{R^2}\right)^{1/4}}{1+\left(1+\frac{1}{R^2}\right)^{1/4}\left(1+\frac{\delta^2}{R^2}\right)^{1/4}}\right)$$

$$+ O\left(\frac{1}{\sqrt{R}}\log\frac{\left(1+\frac{1}{\sqrt{R}}\right)^{1/4}-1}{\left(1+\frac{1}{R^2}\right)^{1/4}+1}\cdot\frac{\left(1+\frac{\delta^2}{R^2}\right)^{1/4}+1}{\left(1+\frac{\delta^2}{R^2}\right)^{1/4}-1}\right)$$

$= o\,(1)$ as $R \to \infty$; $\delta \to 0$ by choosing δ suitably as dependent on R.

Hence (3.3) follows.

Reference

1. E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations*, Vol. I. Oxford University Press, 1962.

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SATAKE DIAGRAMS, IWASAWA AND LANGLANDS DECOMPOSITIONS OF CLASSICAL LIE SUPERALGEBRAS $A(m, n)$, $B(m, n)$ AND $D(m, n)$

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Modified Satake diagrams are obtained for the Lie superalgebras $A(m, n)$, $B(m, n)$ and $D(m, n)$ in order to get the involutive automorphisms of these Lie superalgebras. Iwasawa and Langlands decompositions are done on the same lines as for the ordinary Lie algebras. These decompositions facilitate the determination of the parabolic subalgebras. The parabolic subalgebras can be used to determine the representations using Schmidt constructions.

1. INTRODUCTION

Graded Lie algebras or Lie superalgebras have generated a great interest in particle physics in the contex of supersymmetries. In this paper, we make a preliminary attempt to obtain the induced representations of some Lie superalgebras by the Schmidt construction[1] method. In order to do this, the minimal parabolic subalgebras and other parabolic subalgebras have been found out through which one can get the Lie supergroups corresponding to some Lie superalgebras by induced method. Very little is known on the representations of graded Lie groups except Kostant's novel method[2]. Attempts have been made to obtain typical and atypical representations of some special Lie superalgebras. The analysis of irreducible representations seem to be inadequate[3]. Here, we attempt to present a systematic study of the representations of superalgebras through the prescription subscribed for ordinary simple Lie algebras.

We devise here the super Satake diagrams for the Lie superalgebras on the same lines as for ordinary Lie algebras[4,5], to find out the automorphisms of the Lie superalgebras $A(m, n)$, $B(m, n)$ and $D(m, n)$. The plan of the paper is as follows. In Section 2 following Kac[6,7], we give a brief and quick resume of the classification of classical Lie superalgebras along with the root systems and Dynkin diagrams of Lie superalgebras. In section 3, we give the real forms of classical Lie superalgebras. In section 4, we recapitulate the salient features of Iwasawa and Langlands decompositions for Lie superalgebras in the light of earlier analysis for classical Lie algebras[8-10]. In section 5, we discuss the modified Satake diagrams for classical Lie superalgebras $A(m, n)$, $B(m, n)$ and $D(m, n)$ of physical interest. In section 6, we display the Iwasawa and Langlands decompositions and obtain the parabolic and minimal parabolic subalgebras. From the knowledge of parabolic subalgebras, one can resort to Schmidt construction and derive the corresponding induced representations for superalgebras.

2. Classification of Simple Classical Lie Superalgebras

Here, in this section, we present a quick survey of finite-dimensional classical simple Lie superalgebras over C (complex field), their root systems and Dynkin diagrams analysed by Kac.

Let $V_{\underline{0}} = V \oplus V_{\bar{1}}$ be a z_2-graded space, dim $V_{\underline{0}} = m$, dim $V_{\bar{1}} = n$. Let $l(V)$ or $l(m, n)$ be the vector space of all the $(m+n) \times (m+n)$ matrices, written in block form $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is an arbitrary $m \times m$ matrix, B an arbitrary $m \times n$ matrix, C an arbitrary $n \times m$ matrix and D an arbitrary $n \times n$ matrix. The Lie algebra g_0 of $l(V)$ consists of the diagonal block matrices $\begin{pmatrix} A & O \\ O & D \end{pmatrix}$. The odd subspace g_{T} consists of off-diagonal block matrices $\begin{pmatrix} O & B \\ C & O \end{pmatrix}$. The bracket $< x, x' >$ is commutator of two elements of $l(V)$ if x or x' is an element of $g_{\bar{0}}$ and its an anticommutator if x and x' are elements of $g_{\bar{1}}$.

(a) *The Lie Superalgebras $A(m, n)$*

The special linear graded Lie algebra $sl(m, n)$ consists of block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $TrA = TrD$, which is a graded ideal of $l(m, n)$ of codimension one. Its Lie algebra is $sl(m) \times sl(n) \times gl(1)$, ($gl(1)$ is the trivial one-dimensional Lie algebra). If $n \neq m$ then $sl(m, n)$ is simple.
We get

$$A(m, n) = sl(m+1, n+1) \text{ for } m \neq n, m, n \geqslant 0. \qquad \ldots(2.1)$$

The roots of $A(m, n)$ are expressed in terms of linear functions $\epsilon_1, \ldots, \epsilon_{m+1}, \delta_1 = \epsilon_{m+2}, \ldots, \delta_{m+1} = \epsilon_{m+n+2}$. Even roots are given by $\Delta_0 = \{\epsilon_i - \epsilon_j; \delta_i - \delta_j\}$, $i \neq j$ and the odd roots are $\Delta_1 = \pm(\epsilon_i - \epsilon_j)$. The simplest system of roots is

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{m+\bar{1}}, \delta_1 - \delta_1 - \delta_2, \delta_{n\bar{1}}, \delta_{n+1}\}. \qquad \ldots(2.2)$$

(b) *Orthosymplectic Lie superalgebras*

Suppose that $n = 2r$ is an even positive integer. Let the subalgebra of $l(m, 2r)$ consist of all block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which satisfy

$$t_{AG+GA-0},\ t_{D+D-0},\ C = t_{BG}. \qquad \ldots(2.3)$$

This subalgebra is simple and its Lie algebra is $Sp(2r) \times O(m)$. It is denoted by $osp(m, 2r)$.

(*i*) *Lie superalgebra $B(m, n)$*—By using Cartan's notation

$$B(m, n) = Osp(2m+1, 2n), m \geqslant 0, n > 0. \qquad \ldots(2.4)$$

The roots of $B(m, n)$ are expressed in terms of linear functions $\epsilon_1, \ldots, \epsilon_n, \delta_1 = \epsilon_{2m+1}, \ldots, \delta_n = \epsilon_{2m+n}$. Then even roots Δ_0 are given as

$$\Delta_0 = \{\pm \epsilon_i \pm \epsilon_j;\ \pm 2\delta_i;\ \pm \epsilon_i;\ \pm \delta_i \pm \delta_j\}\ i \neq j;$$

and the odd roots are

$$\Delta_1 = \{\pm \delta_i;\ \pm \epsilon_i \pm \epsilon_j\}.$$

The simplest system of simple roots is given by

$$\{\delta_1 - \delta_2, \ldots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots \epsilon_{m-1} - \epsilon_m, \epsilon_m\} \text{ if } m > 0,$$

and

$$\{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n\} \qquad \ldots(2.5)$$

if $m = 0$.

(*ii*) *Lie superalgebras* $D(m, n)$, $C(n)$—The Lie algebra of $D(m, n)$ is of the type $D_m \oplus C_n$ and it can be written as

$$D(m, n) = Osp(2m, 2n),\ m \geqslant 2, n > 0. \qquad \ldots(2.6)$$

The roots of $D(m, n)$ are expressed in terms of linear functions $\epsilon_1, \ldots, \epsilon_m, \delta_1 = \epsilon_{2m+1}, \ldots, \delta_n = \epsilon_{2m+n}$. Even roots are

$$\Delta_0 = \{\pm \epsilon_i \pm \epsilon_j;\ \pm 2\delta_i;\ \pm \delta_i \pm \delta_j\},\ i \neq j$$

and the odd roots are $\Delta_1 = \{\pm \epsilon_i \pm \epsilon_j\}$. The simplest systems of simple roots are

$$\{\delta_1 - \delta_2, \ldots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m\};$$

$$\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n\}. \qquad \ldots(2.7)$$

The Lie superalgebra $C(n)$ is defined as follows.

The matrices of the Lie algebra of $C(n)$ are of the form

$$\left(\begin{array}{cc|cc} \alpha & 0 & & \\ & -\alpha & & \\ \hline & & a & b \\ & & c & -a^t \end{array}\right)$$

where a, b and c are $(n-1) \times (n-1)$ matrices, b and c being symmetric and $\alpha \in k$,

$$C(n) = Osp(2, 2n-2),\ n \geqslant 2. \qquad \ldots(2.8)$$

The roots of $C(n)$ are expressed in terms of linear functions $\epsilon_1, \delta_1 = \epsilon_3, \ldots, \delta_{n-1} = \epsilon_{n+1}$. Even roots are $\Delta_0 = \{\pm 2\delta_i;\ \pm \delta_i \pm \delta_j\}$; and the odd roots are $\Delta_1 = \{\pm \epsilon_1 \pm \delta_i\}$. Following are the systems of simple roots

$$\pm\{\epsilon_1 - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1}, 2\delta_{n-1}\};$$

$$\pm\{\delta_1 - \delta_2, \ldots, \delta_i - \epsilon_i, \epsilon_i - \delta_{i+1}, \ldots, \delta_{n-2} - \delta_{n-1}, 2\delta_{n-1}\};$$

$$\pm\{\delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1}, \delta_{n-1} - \epsilon_1, \delta_{n-1} + \epsilon_1\}. \qquad \ldots(2.9)$$

(c) *Lie superalgebras* $Q(n)$, $n \geqslant 2$— Define a subalgebra $\widetilde{Q}_n$ of $sl(n+1, n+1)$ by matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where $Tr\, B = 0$. The quotient algebra $\widetilde{Q}_n/z = \overset{\vee}{Q}(n)$ is simple where z is the centre of Q_n.

(d) *Lie superalgebra* $P(n), n \geqslant 2$ —This is also a subalgebra of $sl(n+1, n+1)$ containing matrices of the form $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ where $TrA = 0$, B is a symmetric matrix and C is a skew-symmetric matrix. Its Lie algebra is $sl(n)$.

(e) *The Lie superalgebras* $F(4)$, $G(3)$ *and* $D(2, 1; \alpha)$—There is one and only one 40-dimensional classical Lie superalgebra $F(4)$ for which $F(4)$ is a Lie algebra of type $B_3 \oplus A_1$ and its representation on $F(4)$ is $\text{spin}_7 \otimes sl_2$. The roots are expressed in terms of linear functions $\epsilon_1, \epsilon_2, \epsilon_3$ corresponding to B_3 and δ, corresponding to A_1. Then even roots are $\Delta_0 = \{\pm \epsilon_i \pm \epsilon_j; \pm \epsilon_i; \pm \delta\}$,

$i \neq j$; and odd roots are $\Delta_1 = \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)$.

There are four systems of simple roots :

$$\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), -\epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\};$$

$$\{-\delta, (\tfrac{1}{2}\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), -\epsilon_1, \epsilon_1 - \epsilon_2\};$$

$$\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 - \delta), \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta), \epsilon_1 - \epsilon_2\};$$

$$\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \delta), \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2\}. \qquad \ldots(2.10)$$

There is one and only one 31-dimensional classical Lie superalgebra $G(3)$ for which $G(3)_{\bar{0}}$ is a Lie algebra of type $G_2 \oplus A_1$ and its representation on $G(3)_{\bar{1}}$ is $G_2 \otimes Sl_2$. The roots are expressed in terms of linear functions $\epsilon_1, \epsilon_2, \epsilon_3$, corresponding to G_2, $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, and δ, corresponding to A_1.

The even roots

$$\Delta_0 = \{\epsilon_i - \epsilon_j; \pm \epsilon_i; \pm 2\delta\}$$

and the odd roots are

$$\Delta_1 = \{\pm \epsilon_i \pm \delta; \pm \delta\}.$$

There is a unique system of simple roots

$$\{\delta + \epsilon_1, \epsilon_2, \epsilon_3 - \epsilon_2\}. \qquad \ldots(2.11)$$

There is one parameter family of 17-dimensional Lie superalgebras $D(2, 1; \alpha)$, $\alpha \in k^*/\{0, -1\}$, consisting of all simple Lie superalgebras for which $D(2, 1; \alpha)_{\bar{0}}$ is a Lie algebra of type $A_1 \oplus A_1 \oplus A_1$ and its representation on $D(2, 1; \alpha)_{\bar{1}}$ is $sl_2 \times sl_2 \otimes sl_2$. The roots of $D(2, 1; \alpha)$ are expressed in terms of linear functions ϵ_1, ϵ_2 and ϵ_3.

Even roots are $\Delta_0 = \{\pm 2\epsilon_i\}$ and odd roots are $\{\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}$. There are four systems of simple roots :

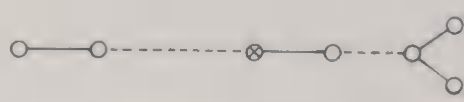
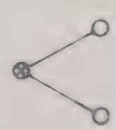
$$\{\epsilon_1 + \epsilon_2 + \epsilon_3, -2\epsilon_i, -2\epsilon_j\}, i \neq j;\ i, j = 1, 2, 3;$$

$$\{\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_1 - \epsilon_2 - \epsilon_3, -\epsilon_1 - \epsilon_2 + \epsilon_3\}. \qquad \text{...(2.12)}$$

After classifying all the classical simple Lie superalgebras, we tabulate the Dynkin diagrams of these superalgebras in Table I. The O, $\otimes$ and ● are called white, grey and black respectively. The white circles imply even roots whereas the grey and

TABLE I

Kac-Dynkin diagrams of Lie superalgebras

Lie Superalgebra	Dynkin Diagram
A(m,n)	
B(m,n)	
B(o,n)	
C(n)	
D(m,n)	
D(2,1,α)	
F(4)	
G(3)	

black circles denote odd roots. The roots are expressed in terms of linear functions $\epsilon_1, \epsilon_2, \ldots, \epsilon_m, \delta_1, \delta_2, \ldots, \delta_n$ which form a unit basis of $\mathbf{h}^*$, the dual space of the Cartan subalgebra $\mathbf{h}$ with inner product $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta_k\ \delta_l) = -\delta_{kl}$, $(\epsilon_i, \delta_k) = 0$ where $1 \leqslant i, j \leqslant m$; $1 \leqslant k, l \leqslant n$. If h_i ($i = 1, 2, \ldots, r$; r = rank of the superalgebra) are the

generators of the Cartan subalgebra and α_i^+ $\left(\alpha_i^-\right)$ are the generators corresponding to the ith positive (negative) simple root, then

$$\left[\alpha_i^+ , \alpha_i^-\right] = \delta_{ij}\, h_i,\ [h_i, h_j] = 0,\ \left[hi,\ \alpha_j^\pm\right] = \pm\, a_{ij}\, \alpha_j^\pm \qquad ...(2.13)$$

where a_{ij} are the elements of the Cartan matrix. The remaining generators may be defined from the simple roots by (anti-) commutation.

3. Reals Forms of Finite-Dimensional Simple Classical Lie Superalgebras[11].

Let $g = g_{\bar{0}} \oplus g_{\bar{1}}$ be a complex classical Lie superalgebra. A real Lie superalgebra g_σ of g is a real form of g if g is the complexification of g_σ. Such a real form g_σ determines a mapping $\sigma : g \to g$. This mapping σ has the following properties :

1. σ is semilinear i. e. $[\sigma X, \sigma Y] = \sigma\, [X, Y]$ for $X, Y \in g$.
2. σ is an involution i. e. $\sigma^2 = I_g$.

The involutive automorphism σ can be written as $\sigma = \sigma_0 + \sigma_1$ where σ_0, σ_1 are its restrictions to $g_{\bar{0}}$ and $g_{\bar{1}}$ respectively. This can be shown that $g_\sigma = g_{\bar{0}\sigma} \oplus g_{\bar{1}\sigma}$ where $g_{\bar{0}\sigma} = \{x + \sigma x \mid x \in g_{\bar{0}}\}$ and $g_{\bar{1}\sigma} = \{x + \sigma x \mid x \in g_{\bar{1}}$. We also see that if σ and σ' are two involutive automorphisms of g then the real forms g_σ and $g_{\sigma'}$ are isomorphic iff there exists an automorphism ϕ of g such that $\sigma' = \phi\sigma\phi^{-1}$. Also every inner automorphism of $g_{\bar{0}}$ extends to an inner automorphism of g.

The real forms of the classical Lie superalgebras can be uniquely determined upto an isomorphism, by the real form $g_{\bar{0}\sigma}$ of the Lie subalgebra $g_{\bar{0}}$. The real forms are listed in Table II from where one can single out the compact real forms of classical Lie superalgebras.

4. Iwasawa and Langlands Decompositions of Simple Classical Lie Superalgebras

Let $g = g_{\bar{0}} \oplus g_{\bar{1}}$ be a complex classical Lie superalgebra. Then there is a compact real form g_σ of g such that

$$g = \mathbf{k} \oplus i\mathbf{p},\ \ g = \mathbf{k} \oplus \mathbf{p}, \qquad ...(4.1)$$

where $\mathbf{k} = (1 + \sigma)\, g_\sigma$ and $\mathbf{p} = i\,(1 - \sigma)\, g_\sigma$, and σ is the involutive automorphism.

Let $\mathbf{h}$ be the Cartan subalgebra of $g_{\bar{0}}$ and Δ the set of positive even and odd roots, $(\Delta_{\bar{0}} \cup \Delta_{\bar{1}})$ of g with respect to $\mathbf{h}$. Following commutation (anti) relations are satisfied by the elements of g,

$[e_\alpha, h] = \alpha\,(h)\, e_\alpha,\ h \in \mathbf{h}$ and $\alpha \in \Delta$.

$[e_\alpha, e_\beta] \neq 0$, if α, β and $\alpha + \beta \in \Delta$

$= 0$, otherwise.

TABLE II

Real forms of Finite-dimentional classical Lie superalgbra

Lie Superalgebra g	Real Form $g_{0\sigma}$
$A(m, n)$	1. $Su^*(m) \oplus Su^*(n) \oplus R$
	2. $Su(p, m-p) \oplus Su(r, n-r) \oplus iR$
	3. $sl(m, R) \oplus Sl(n, R) \oplus R$
	4. $sl(n, C)$, if $m = n$
$B(m, n)$, $D(m, n)$	1. $Sp(m, R) \oplus SO(s, q)$
	2. $Sp(r, s) \oplus SO^*(2p)$
$C(n)$	1. $Sp(n, R) \oplus SO(2)$
	2. $Sp(r, s) \oplus SO(2)$
$B(m, 0)$	1. $Sp(m, R)$
$P(n)$	1. $Su(n)$, if n is even
	2. $Sl(n, R)$
$Q(n)$	1. $Su(p, n-p)$
	2. $Su^*(n)$, if n is even
	3. $Sl(n, R)$
$D(2, 1; \alpha)$	1. $sl(2, R) \oplus sl(2, R) \oplus sl(2, R)$
	2. $su(2) \oplus su(2) \oplus sl(2, R)$ when α is real
	3. $sl(2, R) \oplus sl(2, R)$ when $\alpha + \bar{\alpha} = -1$
$G(3)$	1. $sl(2, R) \oplus G_{2,0}$
	2. $sl(2, R) \oplus G_{2,2}$
$F(4)$	1. $Sl(2, R) \oplus SO(7)$
	2. $Sl(2, R) \oplus SO(3,4)$
	3. $SU(2) \oplus SO(2, 5)$
	4. $SU(2) \oplus SO(1, 6)$

$$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})\, h_\alpha \qquad \text{...(4.2)}$$

where

$$(h_\alpha, h) = \alpha(h), \; h \in \mathbf{h}.$$

We define the Killing forms as

$$B(e_\alpha, e_{-\alpha}) = -1.$$

The compact real form g_σ may be taken to consist of ih_α, for $\alpha = \alpha_1, \ldots, \alpha_l$, l = rank of g together with $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ for every root α of g defined with respect to h.

The basis element ih_α, for $\alpha = \alpha_j$, $j = 1, \ldots, l$ (l = rank of the Lie superalgebra) correspond to eigen value $+1$ while $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ both correspond to eigen value ± 1 ($\exp\{\alpha(h)\} = \pm 1$). $\mathbf{k}$ has a basic consisting of ih_α, together with $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ for all α such that $\exp\{\alpha(h)\} = +1$, while the basis of $\mathbf{p}$ consists of $i(e_\alpha + e_{-\alpha})$ and $(e_\alpha - e_{-\alpha})$ for all α such that $\exp\{\alpha(h)\} = -1$, $\mathbf{a}$ is the maximal abelian subalgebra of $\mathbf{p}$ with dimension m_1, and its basis may be taken to have elements of the form $i(e_\alpha + e_{-\alpha})$. Now choose $\mathbf{m}$, the centralizer of $\mathbf{a}$ in $\mathbf{k}$. Its basis elements are of the form $(e_\alpha + e_{-\alpha})$. The complexification of $(\mathbf{a} \oplus \mathbf{m})$ gives a Cartan subalgebra of $\mathbf{g}$. This second Cartan subalgebra of g, denoted by $\mathbf{h}'$ determines the nilpotent subalgebra $\mathbf{n}$ as follows.

There exists an inner automorphism, V that maps $\mathbf{h}'$ into $\mathbf{h}$ such that $h_j = Vh'_j$ where $V = \prod_\alpha V_\alpha$, $h'_j \in \mathbf{h}'$ and $h_j \in \mathbf{h}$ and the product $\prod_\alpha V_\alpha$ is over all α's constituting $\mathbf{a}$ and $\mathbf{m}$. Let Δ^+ denote the set of positive roots defined with respect to $\mathbf{h}$. Then

$$h_\alpha = \sum_{j=1} b_j(\alpha)\, h_j \text{ and } \alpha \in \Delta^+ \text{ if } b_j(\alpha) > 0. \qquad \ldots(4.3)$$

and if j is the least index, then $b_j(\alpha) \neq 0$.

We divide the positive roots Δ^+ in two classes as follows :

$$p_+ = \{\alpha : \alpha \in \Delta^+, \alpha(h) \neq \alpha(V\sigma V^{-1} h)\}$$

$$p_- = \{\alpha : \alpha \in \Delta^+, \alpha(h) = \alpha(V\sigma V^{-1} h) \ \forall\ h \in \mathbf{h}\}. \qquad \ldots(4.4)$$

Let the subalgebra $\tilde{\mathbf{n}}$ be spanned by elements $V^{-1} e_\alpha$ for $\alpha \in p_+$ and $\mathbf{n} = \tilde{\mathbf{n}} \cap \mathbf{g}$. $\mathbf{n}$ is the nilpotent subalgebra of $\mathbf{g}$. Thus the Iwasawa decomposition of $\mathbf{g}$ is given as

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{n}. \qquad \ldots(4.5)$$

Now we discuss, the Langlands decomposition to obtain the parabolic subalgebras of the Lie superalgebras.

A minimal parabolic subalgebra is defined to be any subalgebra that is conjugate to

$$\mathbf{p}_1 = \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{n}. \qquad \ldots(4.6)$$

Any subalgebra of $\mathbf{g}$ containing a minimal parabolic subalgebra is a general parabolic subalgebra. There exists $2^{|m_1|}$ conjugacy classes of parabolic subalgebras of $\mathbf{g}$ and in each such class there is a standard parabolic subalgebra which can be obtained as follows.

Let Σ be the set of roots for $\mathbf{a}$ and let ψ be the set of positive roots in Σ. Let θ denote the subset of ψ. Let $<\theta>$ denote the set of roots in Σ which arises as linear combinations of roots in θ. We define

$$<\theta>_+ = \Sigma_+ \cap <\theta>, <\theta>_- = \Sigma_- \cap <\theta> \qquad ...(4.7)$$

where Σ_+, Σ_- denote the positive and negative roots in Σ.

Let $n_+(\theta)$, $n_-(\theta)$, $n(\theta)$ denote the subspaces of a corresponding to $<\theta>_+$, $<\theta>_-$ and $\Sigma_+ - <\theta>_+$. Now let us define

$$a_\theta = \{a \in \mathbf{a}, \lambda(a) = 0, \text{ for all } \lambda \in \theta\} \qquad ...(4.8)$$

and let $a(\theta)$ be the orthogonal complement of a_θ in a with respect to the Cartan Killing form. Then

$$p_\theta = m_\theta \oplus a_\theta \oplus n_\theta \qquad ...(4.9)$$

is a parabolic subalgebra of $\mathbf{g}$, where

$$m_\theta = \mathbf{m} \oplus n_+(\theta) \oplus n\ (\theta) \oplus a(\theta). \qquad ...(4.10)$$

A real Cartan subalgebra $\mathbf{h}$ is said to be σ-invariant if

$$\mathbf{h} = (\mathbf{h} \cap \mathbf{k}) \oplus (\mathbf{h} \cap \mathbf{p}). \qquad ...(4.11)$$

A parabolic subalgebra p_θ is said to be cuspidal if there exists a σ-invariant real Cartan subalgebra $\mathbf{h}$ such that

$$a_\theta = \mathbf{h} \cap \mathbf{p}. \qquad ...(4.12)$$

This shows that the minimal parabolic subalgebra is cuspidal.

5. Satake Diagrams[4] and Inner Automorphisms[12] of Classical Lie Superalgebras

Let g_σ be real simple Lie superalgebra and g, which is a complex simple Lie superalgebra, be the complexification of g_σ. Let $\mathbf{h}$ be the Cartan subalgebra and $\mathbf{h}^*$ be the vector space dual of $\mathbf{h}$. Δ is the set of roots of g with respect to $\mathbf{h}$. Let C be the conjugation of g defined by g_σ so that

$$C(x_1 + ix_2) = x_1 - ix_2 \text{ for } x_1, x_2 \in g_\sigma \qquad ...(5.1)$$

C acts on the root space Δ as follows :

for each root $\alpha \in \Delta$, define $\sigma(\alpha)$ by

$$\sigma(\alpha)(h) = \overline{\sigma(C(h))}\ , h \in \mathbf{h} \qquad ...(5.2)$$

then

$$C(g^\alpha) = g^{\sigma}(^\alpha).$$

The mapping $\alpha \to \sigma(\alpha)$ extends by linearity to an involutory isometry of the Euclidean space $\mathbf{h}^*$ under which $\Delta - \Delta_0$ is stable, and Δ_0 is the set of roots $\alpha \in \Delta$, such that[5,12]

$$\alpha = (-1)^{|\alpha|} \sigma(\alpha) \qquad \text{...(5.3)}$$

$|\alpha|$ is equal to zero for an even root and is $+1$ for an odd root.

We have

$$\alpha - (-1)^{|\alpha|} \sigma(\alpha) = \bar{\alpha} \text{ for all } \alpha \in \Delta. \qquad \text{...(5.4)}$$

Let B be the basis of Δ such that $B_0 = B \cap \Delta_0$ is a basis of Δ_0, and such $\Delta^+ - \Delta_0^+$ is σ-stable, where Δ^+ and Δ_0^+ are the set of positive roots determined by B and B_0 respectively. The involution σ determines an involutory permutation of $B - B_0$ as follows : if $\alpha \in B - B_0$ there exists a unique $\beta \in B - B_0$ such that

$$\sigma(\alpha) = \beta, \qquad \text{...(5.5)}$$

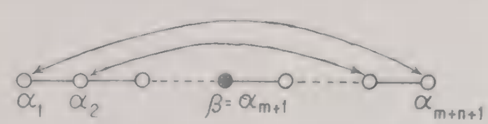
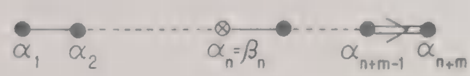
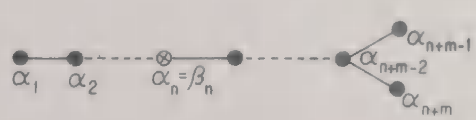
and the mapping $\alpha \rightarrow \beta$ is a permutation of order 2. We have

$$\rho(\alpha) = \rho(\beta) = \tfrac{1}{2}(\alpha + \beta), \text{ where } \rho = \tfrac{1}{2}(1 + \sigma). \qquad \text{...(5.6)}$$

Each real simple Lie superalgebra g_σ determines a normal pair (Δ, σ) which determines g_σ upto isomorphism. A pair (Δ, σ) is said to be normal if $\alpha \in \Delta \Rightarrow \alpha + \sigma(\alpha) \notin \Delta$. The root space Δ may be represented by its Dynkin diagram, the vertices of which represent the elements of the basis B of Δ. The action of σ may be indicated as follows: The vertices of the diagram which represents the elements of B_0 are coloured black and the remainder are coloured white. Two white vertices representing elements $\alpha, \beta \in B - B_0$ for which $\rho(\alpha) = \rho(\beta)$ are joined by an arrow 0 0. The resulting diagram is called the Satake diagrams of g_σ and determines g_σ upto ismorphic Satake diagrams for the Lie superalgebras $A(m, n)$ $m \neq n$, $B(m, n)$ and $D(m, n)$ are given in Table III. The involution σ of Δ is uniquely determined by the Satake diagram.

TABLE III

Satake diagrams for $A(m, n)$, $B(m, n)$ and $D(m, n)$.

Lie Superalgebra	Satake Diagram
$A(m, n)$, $m \neq n$	
$B(m, n)$	
$D(m, n)$	

Let

$$B - B_0 = \{\alpha_1, \ldots, \alpha_r\}$$

and

$$B_0 = \{\beta_1, \ldots, \beta_s\}. \quad \ldots(5.7)$$

Then one can show that[4]

$$-\sigma(\alpha_l) = \alpha_{\pi(l)} + \Sigma(-1)^{|\beta|} n_{ll}\, \beta_l, \ \beta \in \Delta_0 \quad \ldots(5.8)$$

where π is the involutive permutation of 1, 2, ..., r and n_{ll} are non-negative integers.

6. Examples

In this section, we illustrate our analysis for superalgebras of physical interest.

(a) (i) $A(0,1)$—The positive roots are α_1, α_2 and $\alpha_1 + \alpha_2$. $\exp\{\alpha(h)\} = +1$ for α_1 and $\exp\{\alpha(h)\} = -1$ for α_2 and $\alpha_1 + \alpha_2$. Thus $\mathbf{k}$ has a basis ih_α for $\alpha = \alpha_1, \alpha_2$ and $(e_\alpha + e_{-\alpha})$, $i(e_\alpha - e_{-\alpha})$ for $\alpha = \alpha_1$, $\mathbf{p}$ has a basis $i(e_\alpha + e_{-\alpha})$ and $(e_\alpha - e_{-\alpha})$ for $\alpha = \alpha_2, \alpha_1 + \alpha_2$. $\mathbf{a}$ consists of element

$$h_1' = i(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_2 \quad \ldots(6.1)$$

and $\mathbf{m}$ has a basis

$$-ih_2' = e_\alpha + e_{-\alpha}, \ \alpha = \alpha_1. \quad \ldots(6.2)$$

Thus $R_{\mathbf{a}} = \{\alpha_2\}$ and $R_{\mathbf{m}} = \{\alpha_1\}$. Therefore $V = V_\alpha$ with $\alpha = \alpha_2$ and it gives $h_1 = -h_{\alpha_2}$, $h_2 = +h_{\alpha_1}$. Using eqn. (4.3) and eqn. (4.4) we get

$$\Delta^+ = \{\alpha_1, \alpha_1 + \alpha_2, -\alpha_2\} \quad \ldots(6.3)$$

and

$$P_+ = \{\alpha_1, -\alpha_2, \alpha_1 + \alpha_2\}. \quad \ldots(6.4)$$

The basis elements of $\mathbf{n}$ are given by $\mathbf{n} = \tilde{\mathbf{n}} \cap \mathbf{g}$ where the elements of $\tilde{\mathbf{n}}$ are $V^{-1} e_\alpha$ for $\alpha \in p_+$. We see that for this superalgebra we have only two parabolic subalgebras.

(ii) $A(1, 2)$—The positive roots of $A(1, 2)$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. $\exp\{\alpha(h)\} = +1$ for $\alpha = \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$ whereas $\exp\{\alpha(h)\} = -1$ for $\alpha = \alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Therefore, the basis of $\mathbf{k}$ consists of ih_α for $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4$, $(e_\alpha + e_{-\alpha})$, $i(e_\alpha - e_{-\alpha})$ for $\alpha = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ and the basis of $\mathbf{p}$ may be taken to have elements $i(e_\alpha + e_{-\alpha})$ and $(e_\alpha - e_{-\alpha})$ for $\alpha = \{\alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. In the vector space $\mathbf{p}$, $\mathbf{a}$ has basis element

$$h'_1 = i\,(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}. \quad \ldots(6.5a)$$

And the basis of $\mathbf{m}$ then can be taken as

$$-\,ih_j = (e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_1. \quad \ldots(6.5b)$$

The h_i's give

$$\Delta_+ = \{-\alpha_1, \alpha_2, -\alpha_3, \alpha_4, -(\alpha_1+\alpha_2), -(\alpha_3+\alpha_4)$$
$$-(\alpha_1+\alpha_2+\alpha_3), -(\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+\alpha_3),$$
$$-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\} \quad \ldots(6.6)$$

which gives the set p_+

$$= \{-\alpha_1, \alpha_2, -\alpha_3, \alpha_4, -(\alpha_1+\alpha_2), -(\alpha_2+\alpha_3), -(\alpha_3+\alpha_4),$$
$$-(\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\}. \quad \ldots(6.7)$$

The elements of $\mathbf{n}$ are obtained by $V^{-1} e_\alpha$ for $\alpha \in p_+$. We see that for this superalgebra there will be $2^3 = 8$ parabolic subalgebras.

(*b*) $B\,(1,1)$—The positive roots of $g = B\,(1,1)$ are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2$ and $2\alpha_1 + 2\alpha_2$.

From Satake diagram we get $\exp\{\alpha\,(h)\} = +1$, for $\alpha = \alpha_1, \alpha_2$ and $\exp\{\alpha\,(h)\} = -1$ for $\alpha = \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2$. Therefore $\mathbf{k}$ has a basis of ih_α for $\alpha = \alpha_1, \alpha_2$, $e_\alpha + e_{-\alpha}$, $i\,(e_\alpha - e_{-\alpha})$ for $\alpha = \alpha_1, \alpha_2$. $\mathbf{p}$ consists of basis elements $i\,(e_\alpha + e_{-\alpha})$, $(e_\alpha - e_{-\alpha})$ for $\alpha = \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2$. In the vector space $\mathbf{p}$, a the maximal abelian subalgebra has basis element,

$$h'_1 = i\,(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \{2\alpha_1 + 2\alpha_2\}. \quad \ldots(6.8)$$

$\mathbf{m}$ is also one-dimensional with

$$-\,ih'_j = (e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_2. \quad \ldots(6.9)$$

It follows that $V = V_\alpha$ with $\alpha = 2\alpha_1 + 2\alpha_2$ and

$$h_1 = -\left\{\frac{2}{(2\alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2)}\right\}^{\frac{1}{2}} [2h_{\alpha_1} + 2h_{\alpha_2}]$$

and

$$h_2 = -\left\{\frac{2}{(\alpha_2, \alpha_2)}\right\}^{\frac{1}{2}} h_{\alpha_2}. \quad \ldots(6.10)$$

The positive root space Δ^+ is obtained from eqn. (4.3) as

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2\} \quad \ldots(6.11)$$

which gives set p_+ using eqns. (4.4) and (6.11)

$$p_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2\}. \quad \ldots(6.12)$$

The basis elements of $\mathfrak{n}$ are given by

$$\tfrac{1}{2}\{e_{(2\alpha_1+2\alpha_2)} - e_{-(2\alpha_1+2\alpha_2)}\} - \tfrac{1}{2}[2h_{\alpha_1} + 2h_{\alpha_2}];\ \tfrac{1}{2}(e_{\alpha_1} + e_{-\alpha_1})$$

$$+ i\, 2^{-1/2} \operatorname{sgn} N - \alpha_2, \alpha_1 + \alpha_2$$

$$\times (e_{\alpha_1+\alpha_2} + e_{-(\alpha_1+\alpha_2)}) + \tfrac{1}{2} \operatorname{sgn} N_{\alpha_2,\alpha_1} N_{-\alpha_2,+\alpha_2} (e_{\alpha_1+2\alpha_2}$$

$$+ e_{-(\alpha_1+2\alpha_2)});\ \tfrac{1}{2} i (e_{\alpha_1} - e_{-\alpha_1}) - \tfrac{1}{2} \operatorname{sgn} N_{-\alpha_2,\alpha_1+\alpha_2}$$

$$\times (e_{\alpha_1+\alpha_2} - e_{-(\alpha_1+\alpha_2)}) + \tfrac{1}{2} i \operatorname{sgn} N_{\alpha_2,\alpha_1+\alpha_2} N_{-\alpha_2,\alpha_1+\alpha_2} (e_{\alpha_1+2\alpha_2}$$

$$+ e_{-(\alpha_1+2\alpha_2)}),\ \tfrac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - \tfrac{1}{2} i \operatorname{sgn} N_{\alpha_1,\alpha_2} (e_{\alpha_1+\alpha_2} - e_{(\alpha_1+\alpha_2)});$$

$$\tfrac{1}{2} i\ (e_{\alpha_2} + e_{-\alpha_2} + \tfrac{1}{2} \operatorname{sgn} N_{\alpha_1,\alpha_2} (e_{\alpha_1+\alpha_2} + e_{-(\alpha_1+\alpha_2)}). \quad \ldots(6.13)$$

This superalgebra contains $2^1 = 2$ parabolic subalgebras. One is the minimal parabolic subalgebra given by eqns. (6.8), (6.9) and (6.13) as $\mathfrak{p}_1 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and the other being the superalgebra itself.

(c) $B(1,2)$—The positive roots of $\mathfrak{g} = B(1, 2)$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, 2\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3$. From Satake diagram we get

$$\exp\{\alpha(h)\} = +1 \text{ for } \alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3\} \quad \ldots(6.14)$$

and

$$\exp\{\alpha(h)\} = -1 \text{ for } \alpha = \{\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}. \quad \ldots(6.15)$$

$\mathfrak{k}$ has a basis ih_α for $\alpha = \alpha_1, \alpha_2, \alpha_3$, $e_\alpha + e_{-\alpha}$ and $i(e_\alpha - e_{-\alpha})$ for α's given by eqn. (6.14) and $\mathfrak{p}$ has basis elements $(e_\alpha - e_{-\alpha})$ and $i(e_\alpha + e_{-\alpha})$ for α's given by (6.15).

$\mathfrak{a}$ is of dimension one with

$R\mathfrak{a} = \{2\alpha_2 + 2\alpha_3\}$ and the element is given by

$$h_1^{\bar{\theta}} = i(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \{2\alpha_2 + 2\alpha_3\} \quad \ldots(6.16)$$

and $\mathfrak{m}$ is given by

$$-hi_\ell^{\theta} = (e_\alpha + e_{-\alpha}) \text{ for } \alpha = \{\alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3\}. \quad \ldots(6.17)$$

The root space Δ^+ is given as, using eqn. (4.3),

$$\Delta^+ = \{-\alpha_1, \alpha_2, \alpha_3, -(\alpha_1+\alpha_2), \alpha_2+\alpha_3, \alpha_2+2\alpha_3, 2\alpha_2+2\alpha_3, \alpha_1+\alpha_2 + \alpha_3, \alpha_1+\alpha_2+2\alpha_3+2\alpha_2, 2\alpha_3, 2\alpha_1+2\alpha_2+2\alpha_3\}. \quad \ldots(6.18)$$

The subsets p_- and p_+ are given by (using eqn. (4.4))

$$p_- = (\alpha_2+2\alpha_3, 2\alpha_1+2\alpha_2+2\alpha_3\}$$

$$p_+ = \{-\alpha_1, \alpha_2, \alpha_3, -(\alpha_1+\alpha_2), \alpha_2+\alpha_3, 2\alpha_2+2\alpha_3$$
$$\alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+2\alpha_2+2\alpha_3\}. \quad \ldots(6.19)$$

The elements of $\mathbf{n}$ are given by

$$-\tfrac{1}{2}\{e_{2\alpha_2+2\alpha_3} - e_{-(2\alpha_2+2\alpha_3)}\} - \tfrac{1}{2}\, i\,(2h_{\alpha_2}+2h_{\alpha_3}); \frac{1}{\sqrt{2}}(e_{\alpha_1}+e_{-\alpha_1})$$

$$-\frac{i}{\sqrt{2}} \operatorname{sgn} N_{\alpha,\alpha_1}\{e_{\alpha_1+2\alpha_2+2\alpha_3} + e_{-(\alpha_1+2\alpha_2+2\alpha_3)}\}; e_{\alpha_1+\alpha_2} + e_{-(\alpha_1+\alpha_2)};$$

$$\frac{i}{\sqrt{2}}(e_{\alpha_2} - e_{-\alpha_2}) + \frac{1}{\sqrt{2}} \operatorname{sgn} N_{\alpha,\alpha_2}(e_{\alpha_2+2\alpha_3} - e_{-(\alpha_2+2\alpha_3)});$$

$$\frac{1}{\sqrt{2}} \operatorname{sgn} N_{\alpha,\alpha_2+\alpha_3}(e_{\alpha_2+\alpha_3} - e_{-(\alpha_2+\alpha_3)}) + \frac{i}{\sqrt{2}}(e_{\alpha_2+\alpha_3} - e_{-(\alpha_2+\alpha_3)});$$

$$(e_{\alpha_1+\alpha_2+\alpha_3} + e_{-(\alpha_1+\alpha_2+\alpha_3)}); \{e_{\alpha_1+\alpha_2+2\alpha_3} + e_{-(\alpha_1+\alpha_2+2\alpha_3)}\};$$

$$\frac{i}{\sqrt{2}}(e_{\alpha_1+2\alpha_2+2\alpha_3} - e_{-(\alpha_1+2\alpha_2+2\alpha_3)}) + \frac{1}{\sqrt{2}}(\operatorname{sgn} N_{\alpha,\, -(\alpha_1+2\alpha_2+2\alpha_3)})$$
$$(e_{\alpha_1} + e_{-\alpha_1}). \quad \ldots(6.20)$$

The minimal parabolic subalgebras is given by eqns. (6.16), (6.17) and (6.20) as $p_1 = \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{n}$. This superalgebra has got two parabolic subalgebras, the minimal parabolic subalgebras and the Lie superalgebra itself as the dimension of a is one,

(d) $B(2,2)$—The positive roots of $g = B(2, 2)$ are

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_3+2\alpha_4, \alpha_1+\alpha_2+\alpha_3,$$
$$\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+2\alpha_4, 2\alpha_2+2\alpha_3+2\alpha_4,$$
$$\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+\alpha_2+\alpha_3+2\alpha_4, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,$$
$$\alpha_1+2\alpha_2+2\alpha_3, 2\alpha_4+2\alpha_2+2\alpha_3+2\alpha_4.$$

From Satake diagram we have $\exp\{\alpha(h)\} = +1$ for

$$\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2+\alpha_3, \alpha_1+\alpha_2, \alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2 + \alpha_3+\alpha_4, 2\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4\}. \quad \ldots(6.21)$$

and

$$\exp\{\alpha(h)\} = -1 \text{ for } \alpha = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4, \alpha_2 + \alpha_3 + 2\alpha_4, 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}. \quad \ldots(6.22).$$

The basis of **k** is given by ih_α for $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4$, $(e_\alpha + e_{-\alpha})$, $i(e_\alpha - e_{-\alpha})$ for α given by (6.21). Similarly the basis of **p** consists of elements $(e_\alpha - e_{-\alpha})$, $i(e_\alpha + e_{-\alpha})$ for α given by (6.22).

The maximal abelian subalgebra a has one element

$$h_1' = i(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \{2\alpha_2 + 2\alpha_3 + 2\alpha_4\}. \quad \ldots(6.23)$$

The subalgebra **m** is three-dimensional given by

$$-ih_j' = ih_\beta \text{ for } \beta = \{\alpha_3, \alpha_3 + 2\alpha_4, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}. \quad \ldots(6.24)$$

Therefore we can write Δ^+ as, using eqns. (6.23) and (6.24),

$$\Delta^+ = \{-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, \alpha_1 + \alpha_2, -(\alpha_2 + \alpha_3), -(\alpha_3 + \alpha_4), -(\alpha_3 + 2\alpha_4), \alpha_1 + \alpha_2 + \alpha_3, -(\alpha_2 + \alpha_3 + \alpha_4), \alpha_1 + \alpha_2 + \alpha_3 + (\alpha_4, -\alpha_2 + 2\alpha_3 + 2\alpha_4), -(\alpha_2 + \alpha_3 + 2\alpha_4), -(2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\} \quad \ldots(6.25)$$

Subsequently the two subsets p_+ and p_- can be written as follows :

$$p_- = \{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,$$

$$p_+ = \{-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -(\alpha_2 + \alpha_3), -(\alpha_3 + \alpha_4), -(\alpha_3 + 2\alpha_3) - (\alpha_2 + \alpha_3 + \alpha_4), \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \times \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}. \quad \ldots(6.26)$$

The subalgebra $\tilde{\mathbf{n}}$ is characterized by the algebra spanned by elements $V^{-1} e_\alpha$ for all $\alpha \in p_+$ given by eqn. (6.26) and the elements of **n** are defined as $\mathbf{n} = \tilde{\mathbf{n}} \cap \mathbf{g}$. The minimal parabolic subalgebra is given by eqns. (6.23), (6.24) and the elements of **n**. This superalgebra also consists of two parabolic subalgebras only, one the minimal parabolic subalgebra and the other superalgebra itself.

(e) D (2,1)—The positive roots of $\mathbf{g} = D(2, 1)$ are given as $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3$. For this superalgebra

$$\exp\{\alpha(h)\} = +1, \text{ for } \alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2\} \quad \ldots (6.27)$$

and

$$\exp\{\alpha(h)\} = -1 \text{ for } \alpha = \{\alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3\}. \quad \ldots (6.28)$$

The basis of **k** consists of elements ih_α for $\alpha = \alpha_1, \alpha_2$ and $(e_\alpha + e_{-\alpha})$, $i(e_\alpha - e_{-\alpha})$ for α given by eqn. (6.27). And the basis of **p** consists of $(e_\alpha - e_{-\alpha})$, $i(e_\alpha + e_{-\alpha})$ for α givea by eqn. (6.28). The subalgebra **a** is one dimensional and has basis element

$$h_1' = i(e_\alpha + e_{-\alpha}) \text{ with } \alpha = \{2\alpha_1 + \alpha_2 + \alpha_3) \quad (6.29)$$

and **m** is given by

$$ih_r' = ih_\beta \text{ for } \boldsymbol{\beta} = \{\alpha_2, \alpha_3\}. \quad \ldots (6.30)$$

The h_i's give the positive root space Δ^+ as

$$\Delta^+ = \{\alpha_1 - \alpha_2, -\alpha_3, (\alpha_1 + \alpha_2), \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3\}. \quad \ldots (6.31)$$

The two subsets p_- and p_+ are given by

$$p_- = \{\alpha_1, -\alpha_2\},$$

$$p_+ = \{-\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3\}. \quad \ldots (6.32)$$

The elements of $\tilde{\mathbf{n}}$ are given by $V^{-1} e_\alpha$ for $\alpha \in p_+$ and $\mathbf{n} = \tilde{\mathbf{n}} \cap \mathbf{g}$. This superalgebra has also two parabolic subalgebras. one being the minimal parabolic subalgebre and the other being the superalgebra itself.

(f) $D(2.2)$—The positive roots of $\mathbf{g} = D(2,2)$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4, 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The Satake diagrams give

$$\exp\{\alpha(h)\} = 1 \text{ for } \alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4\} \quad \ldots (6.33)$$

and

$$\exp\{\alpha(h)\} = -1 \text{ for } \alpha = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}. \quad \ldots (6.34)$$

The basis of **k** consists of ih_α for $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4, (e_\alpha + e_{-\alpha}), i(e_\alpha - e_{-\alpha})$ for α given by (6. 33) and therefore the basis of **p** is given by $(e_\alpha - e_{-\alpha})$, $i(e_\alpha + e_{-\alpha})$ for α satisfying (6.34). The basis of **a** is

$$h_1' = i(e_\alpha + e_{-\alpha}) \text{ for } \alpha = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \quad \ldots (6.35)$$

and the basis of $\mathfrak{m}$ may be taken containing elements

$$-\, ih'_j = ih_\beta \text{ for } \beta = \{\alpha_3, \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4\}. \qquad ...(6.36)$$

The root space Δ^+ is obtained as

$$\Delta^+ = \{-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -(\alpha_1 + \alpha_2), -(\alpha_2 + \alpha_3) - (\alpha_2 + \alpha_4), -(2\alpha_2 + \alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + \alpha_3), -(\alpha_2 + \alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + \alpha_4), -(2\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 2\alpha_2 + 2\alpha_3 + \alpha_4, -(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)\}. \qquad ..(6.37)$$

The two subsets of Δ^+, p_+ and p_- are given as follows :

$$p_- = \{-(\alpha_1 + \alpha_2), -(\alpha_2 + \alpha_4), -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\} \qquad ..(6.38)$$

$$p_+ = \{-\alpha, -\alpha_2, -\alpha_3, -\alpha_4, -(\alpha_2 + \alpha_3), (2\alpha_2 + \alpha_3 + \alpha_4) - (\alpha_1 + \alpha_2 + \alpha_3), -(\alpha_2 + \alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + \alpha_4), -(2\alpha_1 + \alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)\} \qquad ..(6.39)$$

The elements of $\tilde{\mathfrak{n}}$ are given by $V^{-1} e_\alpha$ for $\alpha \in p^+$ and $\mathfrak{n} = \tilde{\mathfrak{n}} \cap \mathfrak{g}$. This superalgebra will have two parabolic subalgebras, one the minimal parabolic subalgebra and the other the superalgebra itself.

6. Conclusion

We have presented here how to obtain the parabolic and minimal parabolic subalgebras for some Lie superalgebras of physical interest. Schmidt construction, it is hoped, will yield the various representations. We will report more on specific cases in a forthcoming communication.

References

1. R. L. Lipsman, *Lecture Notes in Mathematics*, Vol. 388. Springer, Berlin, 1974.
2. B. Kostant, *Graded Manifolds, Graded Lie Theory and Quantization*, in *Differential Geometric Methods in Physics*, Bonn 1977. (ed. : K. Bleuler and A. Reetz). Lecture Notes on Mathematics, Vol. 570, Springer, Berlin, 1977; B. R. Sitaram and K. C. Tripathy, *J. Math. Phys.* 24 (1983), 164.
3. D. Palev Tchavdar, *J. Math. Phys.* 26 (1985), 1640.
4. O. Loos, *Symmetric Spaces, Vol. II.* Benjamin, New York, 1969.
5. I. G. Macdonald, *Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups*, edited by M. F. Atiyah. Oxford, 91 1977.
6. V. Kac, *Adv. Math.* 26 (1977), 8.
7. M. Scheunert, *Lecture Notes in Mathematics*, vol. 716. Springer-Verlag, New York, 1979.
8. J. F. Cornwell, *J. Math. Phys.* 16 (1975), 1992, 20 (1979), 547.
9. J. F. Cornwell, *Rep. Math. Phys.* 2 (1971), 239, 289.
10. Veena Sharma and K. C. Tripathy, *J. Math. Phys.* 26 (1985), 2485.
11. M. Parker, *J. Math. Phys.* 21 (1980), 689.
12. V. V. Serganova, *Functional Analysis Appl.* 17 (3) (1983), 46; Transl. p. 200; *Math. USSR 'Izvestiya'*, 24 (1985), 539.

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SOME PROPERTIES OF THE SPHERES IN METRIC SPACES

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In a metric space (X, d) if for any distinct points $x, y \in X$ there exist sequences $(s_n) \to y$, $(s'_n) \to x$, $s_n \neq y$, $s'_n \neq x$, $\forall\, n \in N$ with

$$\max \{d(x, s_n), d(y, s_n)\} < d(x, y)$$

and

$$\max \{d(x, s'_n), d(y, s'_n)\} < d(x, y)$$

then the closure of a open sphere is equal to the closed sphere.

Furthermore, if we suppose that for any distinct points $x, y \in X$ there exists $z \in X$ such that $B(x, d(x, y)) \cap B(z, d(z, y)) = \phi$ or the metric space is externally convex[2], the interior of a closed sphere is equal to the open sphere.

§ 1. Let (X, d) be a metric space, $B(x, r) = \{y \in X : d(x, y) < r\}$ the open sphere, $\bar{B}(x, r) = \{y \in X : d(x, y) \leqq r\}$ the closed sphere, cl$A$ the closure and int$A$ the interior of a subset $A \subset X$.

It is well-known[3] that in a normed space the following properties hold :

Property A : for every $x \in X$, $r > 0$, cl $B(x, r) = \bar{B}(x, r)$;

Property B : int $\bar{B}(x, r) = B(x, r)$.

The properties (A) and (B) do not hold in all metric spaces, for example, if (X, d) is a metric space with $d(x, y) = 1$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$, then subsists $B(x, 1) = \{x\}$, $\bar{B}(x, 1) = X$ and Cl $B(x, 1) = \{x\} \neq \bar{B}(x, 1) = X$, int $\bar{B}(x, 1) = X \neq \{x\} = B(x, 1)$.

§ 2. Given any two points α, β, $\alpha \neq \beta$ of the metric space (X, d) we put $E(\alpha, \beta) = \{z \in X : \max \{d(\alpha, z), d(\beta, z)\} < d(\alpha, \beta)\} \cup \{\alpha, \beta\}$ and we denote by $E'(\alpha, \beta)$ the derived set of $E(\alpha, \beta)$, i. e. the set of accmulation points of $E(\alpha, \beta)$.

Lemma 1[1]—For every $x \in X$ and $r > 0$, we have cl $B(x, r) = \bar{B}(x, r)$ if and only if for any $\alpha, \beta \in X$, $\alpha \neq \beta$, $E(\alpha, \beta) \subset E'(\alpha, \beta)$.

Proposition 1—If in a metric space (X, d) the following condition holds : for any distinct points $x, y \in X$ there exist sequence $(s_n) \to y$, $(s'_n) \to x$, $s_n \neq y$, $s'_n \neq x$, $\forall\, n \in N$ with

$$\max \{d(x, s_n), d(y, s_n)\} < d(x, y)$$

and

$$\max \{d(x, s'_n), d(y, s'_n)\} < d(x, y)$$

then for every $x \in X$, $r > 0$, $\mathrm{cl}\, B(x, r) = \bar{B}(x, r)$.

PROOF : According to the lemma we need to show that any distinct points $\alpha, \beta \in X$, $E(\alpha, \beta) \subset E'(\alpha, \beta)$.

Let $z \in E(\alpha, \beta)$, $z \neq \alpha$, $z \neq \beta$ (clearly $\alpha, \beta \in E'(\alpha, \beta)$ by hypothesis).

For the points $\alpha, z \in X$ there exists a sequence $(s_n) \to z$ with

$$\max \{d(\alpha, s_n), d(z, s_n)\} < d(\alpha, z).$$

Observe that for $0 < \epsilon < \min \{d(\alpha, \beta) - d(\alpha, z), d(\alpha, \beta), - d(\beta, z)\}$ exists an integer $n_0 = n_0(\epsilon) \in N$ such that : $\forall\, n \geqq n_0$, $s_n \in B(z, \epsilon)$ and

$$d(s_n, \beta) \leqq d(s_n, z) + d(z, \beta) < \epsilon + d(z, \beta) < d(\alpha, \beta)$$

$$d(s_n, \alpha) \leqq d(s_n, z) + d(z, \alpha) < \epsilon + d(z, \alpha) < d(\alpha, \beta).$$

Hence,

$$s_n \in E(\alpha, \beta), \text{ for } n \geqq n_0. \quad \text{Thus } z \in E'(\alpha, \beta).$$

Remarks 1 : (i) If the metric space (X, d) is convex in the sense of Menger[2] (i.e. for any distinct points $x, y \in X$ there exists at least another point $s \in X$ such that $d(x, s) + d(s, y) = d(x, y)$) and complete, then the hypothesis of Proposition 1 holds (see Theorem 14 1 in Blumenthal[2]).

Generally, if the distance d is convex (i.e. for any distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x, z) = d(y, z) = \frac{d(x, y)}{2}$) the hypothesis of Proposition 1 holds. We note that the convexity of the distance d does non imply that the space X is complete, for example, if $X = (0, 1)$ with $d(x, y) = |x - y|$.

(ii) Let X be the space of the irrational numbers of the open interval $(0, 4)$ and the rational numbers 1, 3 with the distance $d(x, y) = |x - y|$.

It is obvious that the distance d is not convex in X. But, it is easy to prove that the hypothesis of Proposition 1 holds and the space X is neither complete nor connected.

(iii) One verifies easily that every metric subspace (Y, d), $Y \subset X$ of a metric space (X, d) which has the property (A), with Y an open or everywhere dense subset of X, has also property (A).

§ 3 Let X be the closed unit sphere in the plane R^2 with the Euclidean distance d_2. Evidently, for every $x \in X$, $r > 0$, $\bar{B}(x, r) = \text{cl}\, B(x, r)$. In particular, for $x = 0$ (the origin), $r = 1$ we have $\bar{B}(0, 1) = X = \text{cl}\, B(0, 1)$ whereas $\text{int}\, B(0, 1) = X \neq B(0, 1)$.

We can verify that in the space X the hypothesis of Proposition 1 holds (the distance d_2 is convex in X).

Proposition 2—In a metric space (X, d), which satisfies the hypothesis of Proposition 1, if furthermore for any distinct points $x, y \in X$ there exists another $z \in X$ such that

$$B(x, d(x, y)) \cap B(z, d(z, y)) = \phi$$

then for every $x \in X$, $r > 0$ $\text{int}\, \bar{B}(x, r) = B(x, r)$.

Proof : Let $\bar{B}(x, r)$ be any closed sphere. We need to show that no point $y \in X$ with $d(x, y) = r$ is an interior point of the closed sphere $\bar{B}(x, r)$.

In fact, by hypothesis there exists $z \in X$ such that

$$B(x, d(x, y)) \cap B(z, d(z, y)) = \phi.$$

Since the metric space (X, d) satisfies the hypothesis of Proposition 1 there exists a sequence $(s_n) \to y$ with

$$\max\{d(z, s_n), d(y, s_n)\} < d(z, y).$$

We have $d(s_n, z) < (z, y)$ thus $s_n \in B(z, d(z, y))$ and $s_n \notin B(x, d(x, y))$.

Then $s_n \notin \bar{B}(x, d(x, y))$ because, if $d(x, s_n) = d(x, y)$ (for some s_n) then since $s_n \in B(z, d(z, y))$ there exists a neighbourhood $B(s_n, \epsilon)$ contained in $B(z, d(z, y))$ and for the points x, s_n, again by hypothesis of Proposition 1, there exists a sequence $(s'_n) \to s_n$ with

$$\max\{d(x, s'_n),\ d(s_n, s'_n)\} < d(x, s_n)$$

and we would have $s'_n \in B(x, d(x, y)) \cap B(z, d(z, y)$.

But this contradicts the hypothesis.

Hence, in every neighbourhood of the y there exist points which do not belong to $\bar{B}(x, r)$, i.e. $y \notin \text{int}\, \bar{B}(x, r)$.

Definition 2—A metric space (X, d) is externally convex provided it contains for each pair of distinct points $x, y \in X$ at least one point $z \in X$ such that $d(x, y) + d(y, z) = d(x, z)$.

Proposition 3—If a metric space (X, d) satisfies the hypothesis of the Proposition 1 and furthermore, is externally convex then for every $x \in X, r > 0$, $\operatorname{int} \bar{B}(x, r) = B(x, r)$.

PROOF : We will show that no point $y \in X$ with $d(x, y) = r$ is an interior point of the closed sphere $\bar{B}(x, r)$.

Since the metric space is externally convex there exists one point $z \in X$ such that $d(x, y) + d(y, z) = d(x, z)$.

By the hypothesis of Proposition 1, there exists a sequence $(s_n) \to y$ with

$$\max \{d(y, s_n)\}, d(z, s_n)\} < d(y, z), \forall n \in N.$$

We need now to show that $s_n \notin \bar{B}(x, d(x, y))$.

In fact, if $s_n \in \bar{B}(x, r)$ then we would have

$$d(x, s_n) \leqq d(x, y) \Rightarrow d(x, s_n)+d(s_n, z) < d(x, y) + d(y, z) = d(x, z).$$

But this contradicts the triangle inequality.

Remarks 2 : (i) Consider as metric space (X, d) the positive angle of the plane R^2 Oxy including both the positive semiaxis Ox, Oy, with the Euclidean distance.

Observe that in X we have $\operatorname{int} \bar{B}(x, r) = B(x, r)$ whereas the conditions of the Propositions 2 and 3 except for the hypothesis of the Proposition 1, do not hold for the points $x \in Ox$, $y \in Oy$.

Consequently, the conditions of Propositions 2 and 3 are only sufficients.

(ii) One verifies easily that each metric subspace (Y,d), $Y \subset X$ of the metric space (X, d) which has the property (B) by virtue of the conditions of Proposition 2 or 3, with Y an open or everywhere dense subset of X, has also the Property (B).

(iii) Consider as metric space (X, d) the open unit sphere of the plane R^2 with the Euclidean distance. Then the conditions of Propositions 2 and 3 hold, whereas if the space X is the closed unit sphere these do not hold, except the hypothesis of Proposition 1.

REFERENCES

1. N. Aptemiadis, *Math. Revue Greek Math. Soc.* 5 (1977), 47–51.
2. L. Blumenthal, *Distance Geometry*. Oxford, 1953.
3. L. Schwartz, *Analyse : Topologie général et analyse fonctionnelle*. Hermann, Paris, 1970.

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EFFECT OF PULSED LASER ON HUMAN SKIN

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Taking skin as a semi-infinite homogeneous solid with distributed source along a finite depth from the surface, the temperature and thermal stress distributions are obtained, using modified hyperbolic heat conduction equation. The effect of pulsed laser on human skin is considered. Parameters of physical interest are plotted.

1. INTRODUCTION

Lasers find wide use in numerous fields of Science and Engineering. The laser produces radiation with a highly regular light field, outstanding in its high coherence, monochromaticity and directivity. Since laser beams are of high power and can be exploited to produce a targetted effect on material. The applications of lasers include welding, cutting hole burning, isotope separation and medical diogonising etc. Further more, it finds applications in data transmission and processing, measurements and quality control.

The tolerance of the human body to microwaves has been studied by Hendler *et al*[1]. The skin of the human body responds differently for different wavelengths of electromagnetic radiation and as such the spectral characteristics of the laser emission will determine the tolerance.

In this paper, the effect of pulsed laser on human skin, using modified heat conduction equation is considered. The analogous temperature and stress distributions, which includes the effect of finite speed of heat propagation is determined using Laplace transform. The tempearature distribution is computed and exhibited graphically for a ruby laser. The results due to Majumdar[2] can be obtained as a particular case.

2. FORMULATION AND SOLUTION OF THE PROBLEM

The simplest physical model closely resembling normal skin is feasible to mathematical analysis as a translucent and semi-infinite solid. Following the principle of conservation of energy, shorter the pulse duration, the smaller will be the depth of penetration by the radiation, for a given energy input. It is also reasonable to assume

that the thermal inertia of the tissue will prevent any significant change in its thermal and optical constants during the shorter exposure period.

Consider the skin as a semi-infinite homogeneous solid with distributed source along the depth from the surface. Initially, the temperature is considcred to be uniform. The differential equation of heat conduction in this case is

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{h}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{q}{\rho C_v} \qquad \ldots(1)$$

where C is the velocity of propagation of heat, T the temperature distribution, h is the thermal diffusivity, q the rate of energy absorption per unit volume, ρ the density and C_v the specific heat. Equation (1) is obtained from modified heat conduction equation (see Chester[3], Baumeister and Hamill[4]) by considering the finiteness of the heat propagation velocity. The initial and boundary conditions are

at

$$t=0,\ T(x,t)=T_0 \text{ and } \frac{\partial T}{\partial t}=0. \qquad \ldots(2)$$

Here T_0 is the constant temperature.

The regularity boundary condition is $T(\infty, t)=0$ and the effect of penetration gives at

$$x=0 \quad -k\frac{\partial T}{\partial x}=H(1-r) \qquad \ldots(3)$$

where k is the thermal conductivity, H is the constant intensity of the incident radiation and r is the spectral reflectance of the skin for a particular wavelength (λ_1). The right side of equation (3) determines the rate at which energy is absorbed per unit area of the surface and the rate of energy absorbation (q) is

$$q=\alpha H\, e^{-\alpha x}. \qquad \ldots(4)$$

In eqn. (4), α is the absorption coefficient.

Employing the nondimensional variables

$$Z=\frac{C}{2h}\,x,\ \beta=\frac{C^2}{2h}\,t,\ \theta=\frac{T-T_0}{T_0}$$

the heat conduction equation (1), the initial and boundary conditions (2) and (3) can be written as

$$\frac{\partial^2\theta}{\partial\beta^2}+2\,\frac{\partial\theta}{\partial\beta}=\frac{\partial^2\theta}{\partial z^2}+Qe^{-sz} \qquad \ldots(5)$$

at

$$\beta = 0,\ \theta(z, 0) = \frac{\partial \theta}{\partial \beta} = 0,\ \theta(\infty, \beta) = 0 \qquad \ldots(6)$$

at

$$z = 0,\ -\frac{\partial \theta}{\partial z} = \frac{Q(1 - r)}{sh}. \qquad \ldots(7)$$

Here

$$Q = 4h^3 \alpha H / C^2 T_0 k,\ s = 2\alpha h / C.$$

Applying Laplace transform to (5), (6) and (7), one obtains

$$\bar{\theta} = \frac{Q(1 - r)\, e^{-\sqrt{p^2+2p}\,z}}{sh\ p\sqrt{p^2 + 2p}} - \frac{Qs\, e^{-\sqrt{p^2+2p}}}{p\sqrt{p^2+2p}[(p+1)^2 - (1+s^2)]} + \frac{Q\, e^{-sz}}{p[(p+1)^2 - (1+s^2)]} \qquad \ldots(8)$$

where

$\bar{\theta}$ is the temperature in the transformed domain and p is the transform parameter.

The inverse laplace transform of eqn. (8), from Bateman[5], is

$$\theta = A \int_0^\beta g_1(u)\, du - B \int_0^\beta g_2(u)\, g_1(\beta - u)\, du + \frac{Qe^{-sz}}{R} \int_0^\beta e^{-u} \sin h\,(Ru)\, du \qquad \ldots(9)$$

where

$$A = \frac{Q(1 - r)}{sh},\ B = Qs$$

$$g_1(\beta) = 0, \text{ if } \beta < z$$

$$= e^{-\beta} I_0(\sqrt{\beta^2 - z^2}), \text{ if } \beta > z$$

$$g_2(\beta) = \frac{e^{-\beta} \sin h\,(R\beta)}{R}.$$

Here, I_0 is modified Bessel function of zero order and $R^2 = (1 + s^2)$.

The temperature distribution given in eqn. (9) is evaluated numerically.

For the determination of normal thermal stress σ_{xx}, consider the following dynamical equations of themoelasticity.

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots(10)$$

$$\sigma_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \beta_1 T \qquad \ldots(11)$$

where u is the displacement, λ and μ are familiar Lame's constants and $\beta_1 = (3\lambda + 2\mu)\, h$.

From (10) and (11), we get

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{C_1^2} \frac{\partial^2 u}{\partial t^2} = m \frac{\partial T}{\partial x}. \qquad \ldots(12)$$

Here $C_1 = \sqrt{(\lambda + 2u)/\rho}$ is the velocity of propagation of longitudinal wave and $m = \beta_1/(\lambda + 2u)$

Introducting the potential of thermoelastic strain ϕ, where

$$u = \frac{\partial \phi}{\partial x}. \qquad \ldots(13)$$

Using

$U = \frac{C}{2h} u$ and $z = \frac{C}{2h} x$, we get from eqn. (13),

$$U = \frac{\partial \phi^*}{\partial z} \qquad \ldots(14)$$

where

$$\phi^* = \frac{C^2 T_0}{4h^2} \phi.$$

From eqns. (12), (13) and (14), we get

$$\frac{\partial^2 \phi^*}{\partial z^2} - a^2 \frac{\partial^2 \phi^*}{\partial \beta^2} = m\, \theta. \qquad \ldots(15)$$

Here

$$a^2 = C^2/C_1^2 .$$

Applying laplace transform to (15), we obtain

$$\bar{\phi}^* = \frac{mBE}{p^2 (p+2E) [(p+1)^2 - R^2]} \left[\frac{e^{-apz}}{ap} - \frac{e^{-\sqrt{p^2+2p}\, z}}{\sqrt{p^2 + 2p}} \right]$$

$$- \frac{mAE}{p^2 (p + 2E)} \left[\frac{e^{-apz}}{ap} - \frac{e^{-\sqrt{p^2+2p}\, z}}{\sqrt{p^2 + 2p}} \right]$$

$$+ \frac{QS}{a^2 p [(p+1)^2 - R^2](p^2 - b^2)} \left[\frac{e^{-apz}}{ap} - \frac{e^{-sz}}{s} \right]. \qquad ...(16)$$

Here,

$$E = \frac{1}{1 - a^2},\ b = \frac{s}{a}.$$

Following the method of superposition as given in Nowacki[6], the normal stress σ_{zz} is

$$G_{zz} = \tilde{\sigma}_{zz} + \approx{\sigma}_{zz} = \rho \left(\frac{\partial^2 \phi^*}{\partial \beta^2} + \frac{\partial^2 \psi^*}{\partial \beta^2} \right). \qquad ...(17)$$

Here, ψ^* is determined using boundary condition

$\sigma_{zz} = 0$ at $z = 0$ and the equation

$$\frac{\partial^2 \psi^*}{\partial z^2} - a^2 \frac{\partial^2 \psi^*}{\partial \beta^2} = 0. \qquad ...(18)$$

Using Laplace transform, from eqn (17) and (18), we obtain

$$\bar{\psi}^* = -\Bigg[\frac{mBE}{p^2 (p+2E)[(p+1)^2 - R^2]} \left[\frac{1}{ap} - \frac{1}{\sqrt{p^2+2p}} \right]$$

$$- \frac{mAE}{p^2 (p+2E)} \left[\frac{1}{ap} - \frac{1}{\sqrt{p^2+2p}} \right]$$

$$+ \frac{Qs}{a^2 p [(p+1)^2 - R^2](p^2 - b^2)} \left[\frac{1}{ap} - \frac{1}{s} \right]\Bigg] e^{-apz} \qquad ...(19)$$

and

$$\bar{\sigma}_{zz} = \rho\, p^2 [\bar{\phi}^* + \bar{\psi}^*]. \qquad ...(20)$$

Now, from eqn. (16), (19) and (20), we obtain

$$\bar{\sigma}_{zz} = \rho \Bigg[\frac{mBE}{(p+2E)[(p+1)^2 - R^2]\sqrt{p^2+2p}} [e^{-apz} - e^{-\sqrt{p^2+2p}\, z}]$$

$$- \frac{mAE}{(p+2E)} \left[\frac{e^{-apz}}{\sqrt{p^2+2p}} - \frac{e^{-\sqrt{p^2+2p}\, z}}{\sqrt{p^2+2p}} \right]$$

$$+ \frac{Qs}{a^2 p [(p+1)^2 - R^2](p^2 - b^2)} \left[\frac{e^{-apz}}{s} - \frac{e^{-sz}}{s} \right]\Bigg]. \qquad ...(21)$$

From Bateman[5], the inverse transform of eqn. (21) is

$$\sigma_{zz} = \rho\, [mB(F_1 - F_2) - mA(F_3 - F_4) + Q(F_5 - F_6)] \qquad ...(22)$$

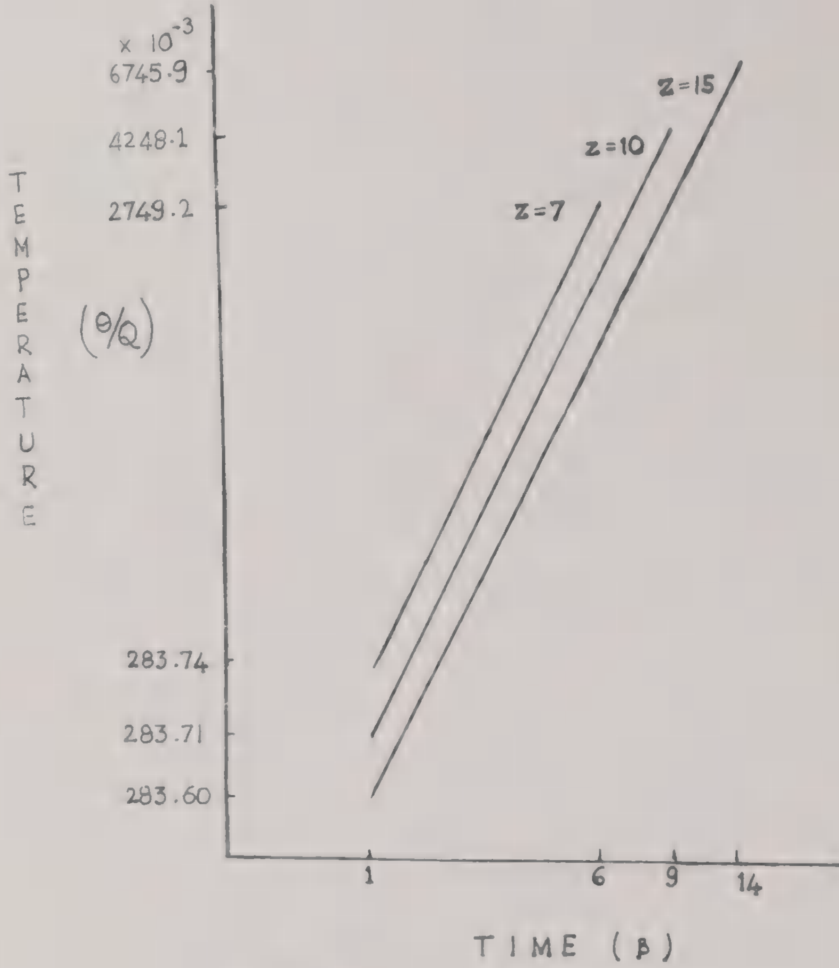

FIG. 1. Temperature *Vs* time.

where

$$F_1 = E \int_0^\beta n^{-2Ey}\, U(y - az)\, f_1(\beta - y)\, dy, \quad F_2 = E \int_0^\beta f_3(y)\, f_4(\beta - y)\, dy,$$

$$F_3 = E \int_0^\beta e^{-2Ey}\, U(y - az)\, f_2(\beta - y)\, dy, \quad F_4 = E \int_0^\beta e^{-2Ey} f_5(\beta - y)\, dy,$$

$$F_5 = \frac{1}{a^2} \int_0^\beta U(y - az)\, f_6(\beta - y)\, dy, \quad F_6 = \frac{e^{-sz}}{a^2} \int_0^\beta f_6(-y)\, dy$$

$$f_1(\beta) = \int_0^\beta f_2(y)\, f_3(\beta - y)\, dy, \quad f_2(\beta) = \frac{1}{\pi} \int_0^\beta \frac{e^{-2y}}{y(\beta - y)}\, dy$$

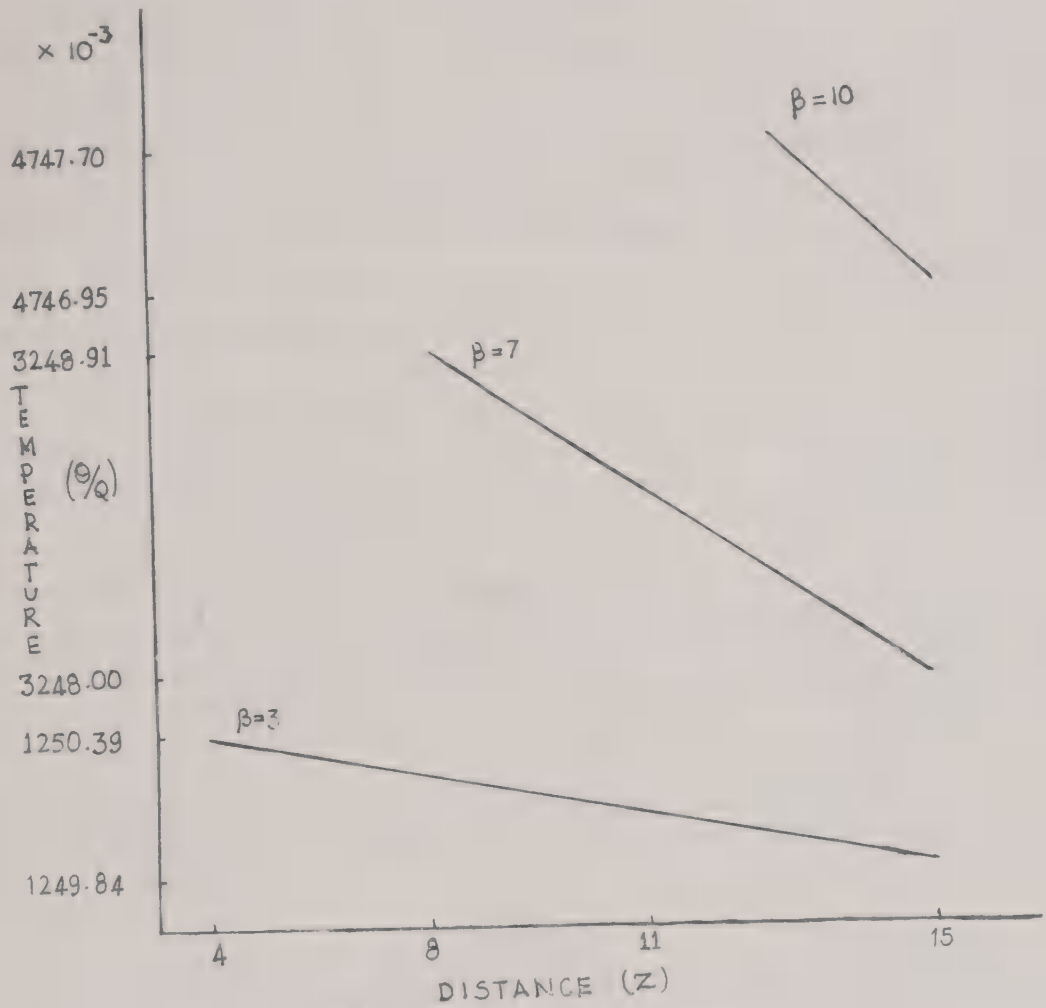

FIG. 2. Temperature *Vs* Distance.

$$f_3(\beta) = \frac{1}{R} e^{-\beta} \sinh(2\beta), \; f_4(\beta) = \int_0^{\beta} f_5(y)\, e^{-2E(\beta-y)}\, dy,$$

$$f_5(\beta) = e^{-\beta} I_0 \left(\sqrt{\beta^2 - z^2}\right) U(\beta - z), \; f_6(\beta) = \frac{1}{b} \int_0^{\beta} \sinh(by)$$

$$f_3(\beta - y)\, dy$$

in the above expressions $U(y - az)$, $U(\beta - z)$ are unit step functions.

3. Results and Discussion

In order to interpret, the temperature variation, the initial temperature of the skin is 33° C, $\rho C_v = 1$, $k = 10^{-3}$ c. g. s. units are taken. The values of optical con-

stants depend on the nature of radiation employed. In addition, for computation, we take ruby laser of wavelength $\lambda_1 = 6943°A$ with the following values for the parameters involved, $H = 0.1$, $r = 40$ and $\alpha = 20$.

The temperature distribution given in eqn. (9) is evaluated numerically for the parameters given above. The variation of temperature distribution at $z = 7$, $z = 10$ and $z = 15$, respectively are presented in Fig. 1, where as in Fig. 2, it is given for $\beta = 3$, $\beta = 7$ and $\beta = 10$.

In Fig. 1, it is seen that the temperature is linear one and increases with time, in general, and as the rate of penetration increases, the temperature will also increase. This phenomena is practicably feasible. In Fig. 2, temperature as a function of distance for different values of time (β) is plotted. As the time increases the temperature will also increase and however for higher values of distance, the temperature keeps on decreasing. As the time increases, the slope of curves decreases.

These results included the effect of finite speed of heat propagation. We can observe the inherent wave nature of the heat transfer process and the temperature at any depth is always less than that of surface temperature during the exposure period. From eqn. (9), the results due to Majumdar[2] can be obtained as a particular case.

Acknowledgement

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References

1. H. Hendler, J. D. Hardy and Dorothy Murgatroyd, *Temperature : Its Measurement and Control in Science and Industry*, Reinhold Publishing Corporation, (1963), p. 211.
2. N. C. Majumdar and V. K. Kochhar, *Def. Sci. J.* 35 (1985), 25.
3. M. Chester, *Phys. Rev.* 131 (1961), 2013.
4. K. J. Baumeister and T. D. Hamill, *J. Heat Transfer, A. S. M. E. Trans.* 91 (1969), 543.
5. Bateman Manuscript Project, Tables of Integral Transforms, Vol. 1, McGraw-Hill, Book Co., Inc., New York, 1954.
7. W. Nowacki, *Bull. Pol. Sci.* 7 (1959), 257.

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THE MODIFIED DINI'S SERIES AND THE FINITE HANKEL-SCHWARTZ INTEGRAL TRANSFORMATION

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In this paper, an arbitrary function $f(x)$ defined on the interval $(0, a)$ is expressed as an expansion in Dini's series of the orthogonal family $\{\mathscr{F}_\nu(\rho_m x)\}$ of modified Bessel functions, where $\mathscr{F}_\nu(x) = x^{-\nu} J_\nu(x)$ and ρ_m denotes the mth postive root of the equation $x\,\mathscr{F}'_\nu(ax) + h\,\mathscr{F}_\nu(ax) = 0$. Next, the convergence theorem is rigorously established. The Dini's series suggest to consider a varaint of the finite Hankel transformation, which will be called the finite Hankel-Schwartz transformation of the second kind. This transformation is used in solving some partial differential equations which cannot be directly treated by applying the corresponding finite Hankel transformation. Finally, we remark that the initial term of our expansion depends only on the parameter h, whereas the classical Dini's expansion depends on $h + \nu$.

1. INTRODUCTION

Schwartz[9] investigated the following modified Hankel transformation

$$F(y) = \int_0^\infty \mathscr{F}(xy) f(x)\, dm(x) \qquad \ldots(1.1)$$

where $dm(x) = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1} dx$ and $\mathscr{F}(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x)$, $J_\nu(x)$ being the Bessel function of the first kind of order ν.

This transformation has been studied in spaces of distributions by several authors and Lee[7] calls it Hankel-Schwartz integral transformation.

To consider the expansion of an arbitrary function $f(x)$ defined in the interval $(0, a)$ as a Fourier-Bessel series, i.e., as a series of the type

$$f(x) = \sum_{n=1}^{\infty} a_n \mathscr{F}_\nu(j_n x) \qquad \ldots(1.2)$$

where $\mathscr{F}_\nu(x) = x^{-\nu} J_\nu(x)$, $\nu \geqq -\frac{1}{2}$ and j_n denote the positive zeros of the functions $\mathscr{F}_\nu(ax)$, i.e.,

$$\mathscr{F}_\nu(j_n a) = 0. \qquad \ldots(1.3)$$

Méndez[8] introduced the corresponding finite transformation through the equation

$$ {}_s h_{1,\nu}\,[f(x)] = F_{1,\nu}(n) = \int_0^a x^{2\nu+1}\,\mathcal{F}_\nu(j_n x) f(x)\,dx \qquad \ldots(1.4)$$

which is called Hankel-Schwartz transformation of the first kind of order ν. Its inversion theorem is stated as :

Theorem 1—Let $f(t)$ be a function defined in (0, 1) and assumed to be absolutely summable over the same interval. Let $\nu \geqq -\frac{1}{2}$ and

$$a_n = \frac{2}{j_n^2\,\mathcal{F}_{\nu+1}^2(j_n)} \int_0^1 t^{2\nu+1}\,\mathcal{F}_\nu(j_n t)\, f(t)\,dt,\ n = 1, 2, \ldots$$

If $f(t)$ is of bounded variation in (a, b), $0 < a < b < 1$, and if $x \in (a, b)$, then the series (1.2) converges to

$$\tfrac{1}{2}\,[f(x+0) + f(x-0)].$$

In this paper we show how the modified Dini series expansion of an arbitrary function $f(x)$ leads naturally to the finite Hankel-Schwartz integral transformation of the second kind. The inverson theorem of this new transformation is rigorously established by studying the convergence of the series expansion. The operational calculus generated is used in the solution of several problems in Mathematical Physics.

Recall that the form of the Dini series is determined by the nature of the zeros of the equation

$$z\,\mathcal{F}'_\nu(z) + h\,\mathcal{F}_\nu(z) = 0.$$

We emphasize that the first term of the expansions depend on the parameter h, and not on $h + \nu$, as it happens in the classical theory (cf. Watson[12], p. 597).

Another interesting feature of this transformation is its usefulness in the solution of partial differential equations which cannot be tackled by applying the corresponding finite Hankel transformation when $\nu \neq 1$ (cf. Colombo[4], p. 82).

Finally, we will denote in the sequel the partial sum of the series (1.2) by

$$S_n(x) = \sum_{m=1}^{n} a_m\,\mathcal{F}_\nu(j_m x). \qquad \ldots(1.5)$$

Setting

$$P_n(x, t) = \sum_{m=1}^{n} \frac{2\,\mathcal{F}_\nu(j_m x)\,\mathcal{F}_\nu(j_m t)}{j_m^2\,\mathcal{F}_{\nu+1}^2(j_m)} \qquad \ldots(1.6)$$

we have

$$S_n(x) = \int_0^1 t^{2\nu+1} P_n(x, t) f(t)\, dt. \qquad \text{...(1.7)}$$

2. Preliminary Results

Let L denote the differential operator $x^{-2\nu-1} \dfrac{d}{dx} x^{2\nu+1} \dfrac{d}{dx}$. We begin by considering the following Sturm-Liouville problem (cf. Sneddon[11], p. 440)

$$(L + \lambda^2)\, y = 0,\ 0 < a < b \qquad \text{...(2.1)}$$

$$My(a) = a_1\, y(a) + a_2\, y'(a) = 0,\ Ny(b) = b_1\, y(b) + b_2\, y'(b) = 0 \qquad \text{...(2.2)}$$

where a_1, a_2, b_1 and b_2 represent prescribed constants.

The general solution of eqn. (2.1) is

$$y = \phi(x, \lambda) = A(\lambda)\, \mathscr{F}_\nu(\lambda x) + B(\lambda)\, \mathscr{Y}_\nu(\lambda x) \qquad \text{...(2.3)}$$

where $\mathscr{F}_\nu(x) = x^{-\nu} J_\nu(x)$ and $\mathscr{Y}_\nu(x) = x^{-\nu} Y_\nu(x)$, $J_\nu(x)$ being the Bessel function of the first kind of order ν and $Y_\nu(x)$ the one of second kind.

Let $y = \phi_n(x)$ be the eigenfunctions of the problem (2.1)—(2.2) which correspond to the nonzero eigenvalues λ_n. We have the general orthogonality condition

$$\int_a^b x^{2\nu+1} \phi_n(x)\, \phi_m(x)\, dx = \begin{cases} \dfrac{1}{2\lambda_n^2}\Big[x^{2\nu+1} \{x\, (\phi_n'(x))^2 + \lambda_n^2\, x \phi_n^2(x) \\ \qquad\qquad + 2\nu\, \phi_n(x)\, \phi_n'(x)\}\Big], & \text{if } m = n \\ 0 & \text{, if } m \neq n. \end{cases} \qquad \text{...(2.4)}$$

Then, we deduce from (2.3) that the solution of the particular problem

$$(L + \lambda^2)\, \phi(x) = 0,\quad 0 \leqslant x \leqslant a \qquad \text{...(2.5)}$$

$$N\phi(a) = \phi'(a) + h\phi(a) = 0,\ h > 0$$

is

$$\phi_n(x) = \mathscr{F}_\nu(\rho_n x) \qquad \text{...(2.6)}$$

where ρ_1, ρ_2, ..denote the positive zeros arranged in ascending order of magnitude of the transcendental equation

$$z\ \mathscr{F}_\nu'(az) + h\ \mathscr{F}_\nu(az) = 0 \qquad \text{...(2.7)}$$

i.e., [cf. Gray *et al.*[6], p. 16, eqn. (24)],

$$-az^2\ \mathscr{F}_{\nu+1}(az) + h\mathscr{F}_\nu(az) = 0.$$

The above orthogonality condition (2.4) now becomes

$$\int_0^a x^{2\nu+1}\, \mathcal{F}_\nu(\rho_m x)\, \mathcal{F}_\nu(\rho_n x)\, dx = \begin{cases} \dfrac{a^{2+\nu}}{2\rho_n^2}(ah^2 + a\rho_n^2 - 2\nu h)\, \mathcal{F}_\nu^2(\rho_n a), & \text{if } m=n \\ 0, & \text{if } m \neq n. \end{cases} \qquad \ldots(2.8)$$

Given an arbitrary function $f(x)$ defined in the interval $(0, a)$, (2.8) allows one to formally express this function as a Dini expansion, as follows

$$f(x) = \sum_{m=1}^{\infty} b_m\, \mathcal{F}_\nu(\rho_m x) \qquad (2.9)$$

where

$$b_m = \frac{2\rho_m^2}{a^{2\nu+1}\, \mathcal{F}_\nu^2(\rho_m a)\,(ah^2 + a\rho_m^2 - 2\nu h)} \int_0^a x^{2\nu+1}\, \mathcal{F}_\nu(\rho_m x)\, f(x)\, dx \qquad \ldots(2.10)$$

$m = 1, 2, \ldots$, ρ_m being the positive roots of eqn. (2.7).

Note that we can extend the Dixon theorem (cf. Watson[12], p. 480) to the zeros of the function (2.7). Indeed, it can be proved that the zeros of the equations $Ax\, \mathcal{F}'_\nu(x) + B\mathcal{F}_\nu(x) = 0$ and $Cx\, \mathcal{F}'_\nu(x) + D\mathcal{F}_\nu(x) = 0$ are interlaced, whatever the real numbers A, B, C and D, provided they are such that $AD \neq BC$. Hence, the roots of eqns. (1.3) and (2.7) also are interlaced.

3. The Modified Dini Expansion—The Convergence Theorem

In this section it will be assumed that $a = 1$ for the sake of simplicity.

As it occurs in the classical theory of Dini expansions (cf. Watson[12], p. 597), we have to add an initial term to the series (2.9). In fact, the form of Dini expansion is based on the zeros of the function (2.7) and these depend upon the values of the parameter h. Thus, the expansion (2.9) only corresponds to the case $h > 0$.

When $h = 0$ it can easily be seen that the equation (2.7) has a zero at the origin. On the other hand, when $h < 0$ this function has two purely imaginary zeros.

Let ν be a real number such that $\nu \geqslant -\frac{1}{2}$. We write the modified Dini expansion of $f(x)$, as follows

$$f(x) = b_0 + \sum_{m=1}^{\infty} b_m\, \mathcal{F}_\nu(\rho_m x) \qquad \ldots(3.1)$$

where b_0 denote the initial term which must be inserted in (2.9) as a consequence of the existence of these new roots.

If $h > 0$ the initial term $b_0 = 0$ and (3.1) coincides with (2.9). But when $h = 0$, taking into account that

$$\int_0^1 x^{2\nu+1} \mathcal{F}_\nu (\rho_m x)\, dx = \mathcal{F}_{\nu+1} (\rho_m) = 0,\ m = 1, 2, \ldots$$

[cf. Gray *et al.*[6], p. 16, eqn. (25)] and (2.7), we get

$$b_0 = (2\nu + 2) \int_0^1 x^{2\nu+1} f(x)\, dx. \qquad \ldots(3.2)$$

Finally, if $\pm \rho_0 i$ denote the imaginary zeros of (2.7) when $h < 0$, from (3.1) and (2.8) we infer that

$$b_0 = \frac{2\rho_0^2}{(\rho_0^2 + 2\nu h - h^2)\, \mathcal{F}_\nu^2 (\rho_0 i)} \int_0^1 x^{2\nu+1} \mathcal{F}_\nu (i\rho_0 x) f(x)\, dx. \qquad \ldots(3.3)$$

Now, consider the function

$$\frac{2w\, \mathcal{F}_\nu (xw)\, \mathcal{F}_\nu (tw)}{\mathcal{F}_\nu (w) \{w \mathcal{F}'_\nu (w) + h\, \mathcal{F}_\nu (w)\}} \qquad \ldots(3.4)$$

whose poles are the zeros $j_1, j_2, \ldots$ of $\mathcal{F}_\nu (z)$ and the zeros $\rho_1, \rho_2 \ldots$ of $z\, \mathcal{F}'_\nu (z) + h\, \mathcal{F}_\nu (z)$.

The residues of this function at the first poles are

$$\frac{2\, \mathcal{F}_\nu (j_m x)\, \mathcal{F}_\nu (j_m t)}{j_m^2\ \mathcal{F}_{\nu+2}^2 (j_m)}.$$

If $h > 0$ the residues at the poles $\rho_1, \rho_2, \ldots$ are

$$-\frac{2\rho_n^2\ \mathcal{F}_\nu (\rho_n x)\, \mathcal{F}_\nu (\rho_n t)}{(\rho_n^2 - 2\nu h + h^2)\, \mathcal{F}_\nu^2 (\rho_n)}.$$

When $h = 0$ we must moreover consider the residue at the origin, whose value is $-4(\nu + 1)$.

When $h < 0$ the residues at $\pm i\rho_0$ are both equal to

$$-\frac{2\rho_0^2\ \mathcal{F}_\nu (\rho_0 xi)\, \mathcal{F}_\nu (\rho_0 ti)}{(\rho_0^2 + 2\nu h - h^2)\, \mathcal{F}_\nu^2 (\rho_0 i)}.$$

By denoting the partial sum of the series (3.1)

$$\sigma_n(x) = b_0 + \sum_{m=1}^{\infty} b_m \mathscr{F}_\nu(\rho_m x) \qquad \ldots(3.5)$$

and

$$P_n(x, t; h) = A_0(x, t) + \sum_{m=1}^{n} \frac{2\rho_m^2 \mathscr{F}_\nu(\rho_m x) f_\nu(\rho_m t)}{(\rho_m^2 - 2\nu h + h^2)\mathscr{F}_\nu^2(\rho_m)} \qquad \ldots(3.6)$$

where

$$A_0(x, t) = \begin{cases} 0, \text{ if } h > 0 \\ 2(\nu + 1), \text{ if } h = 0 \\ \dfrac{2\rho_0^2 \mathscr{F}_\nu(\rho_0 xi)\mathscr{F}_\nu(\rho_0 ti)}{(\rho_0^2 + 2\nu h - h^2)\mathscr{F}_0^2(\rho_0 i)}, \text{ if } h < 0 \end{cases} \qquad \ldots(3.7)$$

we can express (3.5) as

$$\sigma_n(x) = \int_0^1 t^{2\nu+1} P_n(x, t; h) f(t)\, dt. \qquad \ldots(3.8)$$

Now choose D_n such that it is not equal to any of the number j_m and $\rho_n < D_n < \rho_{n+1}$ and let j_N be the greatest of the numbers j_m which does not exceed D_n (cf. Watson[12], p. 598).

The following expression

$$S_n(x, t; h) = \sum_{m=1}^{n} \frac{2\mathscr{F}_\nu(j_m x)\mathscr{F}_\nu(j_m t)}{j_m^2 \mathscr{F}_{\nu+1}^2(j_m)} - A_0(x, t)$$

$$- \sum_{m=1}^{n} \frac{2\rho_m^2 \mathscr{F}_\nu(\rho_m x)\mathscr{F}_\nu(\rho_m t)}{(\rho_m^2 + h^2 - 2\nu h)\mathscr{F}_\nu^2(\rho_m)} \qquad \ldots(3\ 9)$$

where $A_0(x, t)$ is given by (3.7), permits to connect the partial sums of the modified series of Fourier-Bessel (1.2) and Dini (3.1). Clearly, from (1.5), (1.6), (1.7), (3.5), (3.8) and (3.9) it can be deduced that

$$\int_0^1 t^{2\nu+1} S_n(x, t; h) f(t)\, dt = \sum_{m=1}^{N} a_m \mathscr{F}_\nu(j_m x) - b_0 - \sum_{m=1}^{n} b_m \mathscr{F}_\nu(\rho_m x)$$

$$= S_N(x) - \sigma_n(x). \qquad \ldots(3.10)$$

From Cauchy's theory of residues we find the following integral representation of (3.9)

$$S_n(x, t; h) = \frac{1}{2\pi i} \int_{D_n - \infty i}^{D_n + \infty i} \frac{2w\, \mathscr{F}_\nu(xw)\, \mathscr{F}_\nu(tw)}{\mathscr{F}_\nu(w)\, \{w\, \mathscr{F}_\nu'(w) + h\, \mathscr{F}_\nu(w)\}}\, dw. \quad \ldots(3.11)$$

Since

$$\int_0^t wt^{2\nu+1}\, \mathscr{F}_\nu(tw)\, dt = t^{\nu+2}\, \mathscr{F}_{\nu+1}(tw)$$

[cf. Gray *et al.*[6], p. 16, eqn. (25)], it can be inferred from (3.11) that

$$\int_0^t t^{2\nu+1}\, S_n(x, t; h)\, dt = \frac{t^{2\nu+2}}{\pi i} \int_{D_n - \infty i}^{D_n + \infty i} \frac{w\, \mathscr{F}_\nu(xw)\, \mathscr{F}_{\nu+1}(tw)}{\mathscr{F}_\nu(w)\, \{w\, \mathscr{F}_\nu'(w) + h\, \mathscr{F}_\nu(w)\}}\, dw. \quad \ldots(3.12)$$

As an immediate consequence of (3.11) and (3.12) we have

$$|\, S_n(x, t, h)\, | < \frac{c_3}{(xt)^{\nu+1/2}\, (2 - x - t)} \quad \ldots(3.13)$$

and

$$\left| \int_0^t t^{2\nu+1}\, S_n(x, t; h)\, dt \right| < \frac{c_4}{D_n} \left(\frac{t}{x} \right)^{\nu+1/2} \frac{1}{(2 - x - t)} \quad \ldots(3.14)$$

where c_3 and c_4 are constants independent of n, x and t.

Next, it can be proved with an argument similar to the one used in Watson[12] (p. 599) that if $f(t)$ is absolutely summable in the interval (a, b), $0 \leqslant a < b \leqslant 1$, then

$$\int_a^b t^{2\nu+1}\, S_n(x, t; h)\, f(t)\, dt \to 0, \text{ as } n \to \infty \quad \ldots(3.15)$$

provided $0 < x < 1$.

Theorem 2—Let $f(t)$ be a function defined and absolutely summable in the interval (0, 1). If $f(t)$ is of bounded variation in (a, b) where $0 \leqslant a < b \leqslant 1$, then the series (3.1) converges to the sum $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at all points x such that $a + \Delta \leqslant x \leqslant b - \Delta$, $\Delta > 0$ being arbitrarily small.

PROOF : By virture of Theorem 1 the series $\sum_{n=1}^{\infty} a_n\, \mathscr{F}_\nu(j_n x)$ converges to the sum $\frac{1}{2}[f(x + 0) + f(x - 0)]$. Our assertion follows directly from (3.10) and (3.15) to pass to the limit as $n \to \infty$.

Remark 1 : Note that the initial term b_0 of our expansion (3.1) only depends on the value of h, whereas this term depends on $h + \nu$ in the classical theory (cf. Watson[12], p. 598). Moreover, the roots ρ_n of the equation $z\ \mathcal{F}_\nu'(z) + h\ \mathcal{F}_\nu(z) = 0$ are not equal to the roots λ_n's of $z\ J_\nu'(z) + hJ_\nu(z) = (z) = 0$.

4. The Finite Hankel—Schwartz Integral Transformation of the Second Kind—Applications

According to (2.9) and (2.10), we define the finite Hankel-Schwartz integral transformation of the second kind of order $\nu > -\frac{1}{2}$ by the equation

$$ {}_sh_{2,\nu}[f(x)] = F_{2,\nu}(n) = \int_0^a x^{2\nu+1}\, \mathcal{F}_\nu(\rho_n x) f(x)\, dx \qquad \text{...(4.1)}$$

whose kernel is the modified Bessel function (2.6) and where ρ_n denote the roots of eqn. (2.7).

The corresponding inversion formula is

$$ {}_sh_{2,\nu}^{-1}[F_{2,\nu}(n)] = f(x) = b_0 + \frac{2}{a^{2\nu+1}} \sum_{n=1}^{\infty} \frac{\rho_n^2\, F_{2,\nu}(n)\, \mathcal{F}_\nu(\rho_n x)}{(ah^2 + a\rho_n^2 - 2\nu h)\, \mathcal{F}_\nu^2(\rho_n a)} . \qquad \text{...(4.2)}$$

Theorem 2 not only guarantees existence of (4.1) but also ensures that inversion formula (4.2) holds.

If we assume that the $f \in C^2(0, a)$, $f'(a) + h f(a) = 0$ and $h > 0$, we obtain the main operational formula of this transformation, i. e.,

$$ {}_sh_{2,\nu}\left[f''(x) + \frac{2\nu+1}{x} f'(x)\right] = -\rho_n^2\ {}_sh_{2,\nu}[f(x)] \qquad \text{...(4.3)}$$

whatever the values of $f(0)$, and $f'(0)$, provided they are finite.

If $f'(a) + hf(a) \neq 0$ and $h > 0$, we get

$$ {}_sh_{2,\nu}\left[f''(x) + \frac{2\nu+1}{x} f'(x)\right] = a^{2\nu+1}\, \mathcal{F}_\nu(\rho_n a)\,(f'(a) + h f(a)) $$
$$ - \rho_n^2\ {}_sh_{2,\nu}[f(x)]. \qquad \text{...(4.4)}$$

Remark 2 : Recall that the function $y = \mathcal{F}_\nu(x)$ is a solution of the equation $Ly \equiv y'' + \frac{1+2\nu}{x} y' + y = 0$. Its multiplication by $x^{2\nu}$ has only repercussions on the sign of the parameter ν, that is, the function

$$ y = \mathcal{F}_\nu^*(x) = x^{2\nu}\, \mathcal{F}_\nu(x) $$

is a solution of the equation

$$L^* y \equiv y'' + \frac{1-2\nu}{x} y' + y = 0.$$

Consequently, the solution of Sturm-Liouville problem

$$(L^* + \lambda^2)\,\phi(x) = 0$$

$$N^* \phi(a) \equiv \phi'(a) + \left(h - \frac{2\nu}{a}\right)\phi(a) = 0$$

is

$$\phi_n^*(x) = \mathscr{F}_\nu^*\, \rho_n(x) \qquad \ldots(4.5)$$

where $\rho_1, \rho_2, \ldots$ denote the positive zeros of the equation (2.7). The solutions (4.5) form a orthogonal system on the interval $(0, a)$ with respect to the weight function $x^{1-2\nu}$. Proceeding as before, we can now introduce the integral transform

$$_s h^*_{\nu,2}[f(x)] = F^*_{2,\nu}(n) = \int_0^a x^{1-2\nu}\, \mathscr{F}_\nu^*(\rho_n x) f(x)\, dx \qquad \ldots(4.6)$$

whose inversion formula, in the case $h > 0$, is given by

$$_s h^{*-1}_{2,\nu}[F^*_{2,\nu}(n)] = f(x) = \sum_{n=1}^{\infty} \frac{2\rho_n^2\, F^*_{2,\nu}(n)\, \mathscr{F}_\nu^*(\rho_n x)}{a^{1-2\nu}(ah^2 + a\rho_n^2 - 2\nu h)\, \mathscr{F}_\nu^{*2}(\rho_n a)}. \qquad \ldots(4.7)$$

A similar result to that proven in Theorem 2 can be stated in relation with the convergence of the series (4.7), whenever $\nu \geqq -\frac{1}{2}$.

The main operational rule of the transform (4 6) is

$$_s h^*_{2,\nu}\left[f''(x) + \frac{1-2\nu}{x} f'(x) \right] = -\rho_n^2\; {_s h^*_{2,\nu}}[f(x)] \qquad \ldots(4.8)$$

provided that $f'(a) + \left(h - \frac{2\nu}{a}\right) f(a) = 0$ and $h > 0$.

In the sequel we shall give a few examples to illustrate the use of the above transformations in solving some important problems.

(a) Let ν be any real number. We wish to find the solution of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2\nu+1}{r}\frac{\partial u}{\partial r} = \frac{1}{k}\frac{\partial u}{\partial t} \quad (t > 0,\ 0 < r < a,\ k > 0) \qquad \ldots(4.9)$$

satisfying the initial condition

$$u(r, 0) = f(r) \quad (0 \leqq r \leqq a)$$

and the boundary conditions

$$\frac{\partial u(a,t)}{\partial r} + h\,u(a,t) = 0\,,\ \text{for every } \nu \geqq 0,$$

or

$$\frac{\partial u(a,t)}{\partial r} + \left(h - \frac{2\nu}{a}\right) u(a,t) = 0,\ \text{for every } \nu \geqq 0.$$

By virtue of (4.3) and (4.8), we convert formally (4.9) into

$$\left(\frac{\partial}{\partial t} + k\,\rho_n^2\right) U_{n,\nu}(t) = 0,$$

where

$$U_{n,\nu}(t) = \begin{cases} {}_s h_{2,\nu}\,[t\,(r,u)]\,, & \nu \geqq 0 \\ {}_s h^*_{2,-\nu}\,[u\,(r,t)], & \nu \leqq 0. \end{cases}$$

Hence,

$$U_{n,\nu}(t) = F_{2,\nu}(n)\, e^{-k\rho_n^2 t} \qquad \dots (4.10)$$

where

$$F_{2,\nu}(n) = \begin{cases} {}_s h_{2,\nu}\,[f(r)]\,, & \nu \geqq 0 \\ {}_s h^*_{2,-\nu}\,[f(r)] & \nu \leqq 0. \end{cases}$$

By applying the inversion formulas (4.2) and (4.7) to (4.10), we get the required solution

$$u(r,t) = \begin{cases} \dfrac{2}{a^{2\nu+1}} \displaystyle\sum_{n=1}^{\infty} \dfrac{\rho_n^2\, F_{2,\nu}(n)\, \mathscr{F}_\nu(\rho_n r)\, e^{-k\rho_n^2 t}}{(ah^2 + a\rho_n^2 - 2\nu h) + \mathscr{F}_\nu^2\,(\rho_n a)}, & \nu \geqq 0 \\[2ex] \dfrac{2}{a^{2\nu+1}} \displaystyle\sum_{n=1}^{\infty} \dfrac{\rho_n^2\, F_{2,\nu}(n)\, \mathscr{F}^*_{-\nu}(\rho_n r)\, e^{-k\rho_n^2 t}}{(ah^2 + a\rho_n^2 + 2\nu h)\, \mathscr{F}^{*2}_{-\nu}\,(\rho_n a)}, & \nu \leqq 0. \end{cases} \qquad \dots (4.11)$$

Note that when $\nu = 0$ the prablem (4.9) reduces to the one considered by Sneddon[11] (eqns. 8-4-20, 33, 34) on the diffusion equation, since in this case $\mathscr{F}_0(x) = J_0(x)$ and $\rho_n = \xi_n$ are the roots of $x\,J_0'(ax) + h\,J_0(ax) = 0$. Then, both of formulas in (4.11) yield the sum solution and this coincides with the one achieved in the reference mentioned obove.

A procedure similar to the one used by Churchill[3] (p. 191), allows one to establish (4.11) as a rigorous solution of our problem.

Remark 3 : Note that the equation (4.9) cannot be solved directly by means of the finite Hankel transformation, except when $\nu = 0$. Nevertheless, the simultaneous application of the finite Hankel-Schwartz transformations (4.1) and (4.6) provides a simple method to solve immediately the problem (a), no matter what the real value of ν may be.

(b) Many partial differential equations involving the n-dimensional laplacian operator can also be solved by using the transformation (4.1). Indeed, the n-dimensional potential equation is

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_{n-1}^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{.. (4.12)}$$

where $u = u\,(x_1, x_2, \ldots, x_{n-1}, z)$. If we seek solntions which only depend on $r = x^2 + (x_2^2 + \ldots + x_{n-1}^2)^{1/2}$ and z, (4.12) reduces to the from (cf. Sneddon[11], p. 342)

$$\frac{\partial^2 u}{\partial r^2} + \frac{n-2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \text{...(4.13)}$$

We find now the solution of (4.13) that satisfies the conditions

$$\begin{aligned} &\frac{\partial u\,(a, z)}{\partial r} + hu\,(a\, z) = 0 \; (z \geqslant 0, h > 0) \\ &u\,(r, 0) = f\,(r) \qquad \text{...(4.14)} \\ &u\,(r, z) \to 0, \text{ as } z \to \infty, \end{aligned}$$

by directly applying to (4.13) the finite Hankel-Schwartz transformation of the second kind of order $\nu = (n-3)/2$. Now denote $U_n\,(z) = {}_s h_{2,\nu}\,[u\,(r, z)]$. From (4.3) we see that $U_n\,(z)$ satisfies the equation

$$-\,\rho_n^2\; U_n(z) + \frac{\partial^2\, U_n\,(z)}{\partial z^2} = 0$$

whose solution is, in view of conditions (4.14),

$$U_n\,(z) = F\,(n)\, e^{-\rho_n z}$$

where

$$F\,(n) = {}_s h_{2,\nu}\,[f\,(r)].$$

Again making use of (4.2), the formal solution of the problem posed by equations (4.13)—(4.14) is

$$u\,(r, z) = \frac{2}{a^{2\nu+1}} \sum_{n-1}^{\infty} \frac{\rho_n^2\; e^{-\rho_n z}\; \mathscr{F}_\nu\,(\rho_n\, r)\; F\,(n)}{(ah^2 + a\rho_n^2 - 2\nu h)\; \mathscr{F}_\nu^2\,(\rho_n\, a)}. \qquad \text{...(4.15)}$$

That (4.15) is truly a solution of our problem can be proved assuming that the function $f(r)$ is such that the above series and the series obtained by applying the operator L and $\frac{\partial^2}{\partial z^2}$ converge adequately.

When $\nu = 0$ (i.e., $n = 3$) the problem (4.13) consists of finding the bounded steady temperatures $u(r, z)$ in the cylinder $r \leqslant a$, $z \geqslant 0$, if it is assumed that heat transfer into surroundings at temperature zero takes place through the surface $r = a$, according to the linear law $u_r(a, z) = -hu(a, z)$.

Remark 4 : The problem (4.13) is usually solved by means of the finite Hankel transform only in the case $n = 3$ (Colombo[4], p. 82). Now, by combining the finite transforms (4.1) and (4.6), it is feasible to solve this problem for each $n \geqq 3$, even more, for an arbitrary integer n.

References

1. T. M. Apostol, *Mathematical Analysis*, Addision-Wesley, Mass., 1957.
2. H. S. Carslaw, *An Introduction to the Theory of Fourier's Series and Integrals*, Dover Publication, New York, 1950.
3. R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill Book Co., Inc., New York, 1963.
4. S. Colombo. *Les transformations de Mellin et Hankel*, C. N. R. S., Paris, 1959.
5. L. S. Dube and J. N. Pandey, *Tôhoku Math. J.* 27 (1975), 337–54.
6. A. Gray, G. B. Mathews and T. M. MacRobert, *A Treatise on Bessel Functions and their Applications to Physics*, McMillan and Co. Ltd., London, 1952.
7. W. Y. Lee, *SIAM J. Math. Anal.* 6 (2) (1975), 427–32.
8. J. M. Méndez, The finite Hankel-Schwartz transform (To appear) .
9. A. L. Schwartz, *Proc. Am. Math. Soc.* 22 (1969), 713–17.
10. I. N. Snoddon, *Phill. Mag. Ser.* 7, 37 (264) (1964), 17–25.
11. I. N. Sneddon, *The use of Integral Transforms*, Tata-McGraw-Hill, New Delhi, 1979.
12. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1966.

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L^1-CONVERGENCE OF A MODIFIED COSINE SUM

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We introduce here a new modified cosine sum and study its L^1-convergence to a coine trigonometric series belonging to the class S of Sidon[5]

1. INTRODUCTION

Let

$$\frac{a_0}{2} + \sum_{k-1}^{\infty} a_k \cos kx \qquad \ldots(1.1)$$

be a cosine series satisfying $a_k = o(1), k \to \infty$. If there exists a sequence $<A_k>$ such that

$$A_k \downarrow 0, k \to \infty \qquad \ldots(1.2)$$

$$\sum_{k=0}^{\infty} A_k < \infty \qquad \ldots(1.3)$$

$$| \Delta a_k | \leqslant A_k \forall k \qquad \ldots(1.4)$$

we say that (1.1) belongs to the class S introduced by Sidon[5].

Let the partial sum of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Concerning the L^1-convergence of Rees-Stanojević cosine sums[4]

$$f_n(x) = \tfrac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_k \cos kx$$

to a cosine trigonometric series, belonging to the class S, Ram[3] proved the following theorem :

Theorem A—If (1.1) belongs to the class S, then

$$\|f - f_n\|_{L^1} = o(1), n \to \infty.$$

In the present paper, we introduce a new modified cosine sum as

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

and study its L^1-convergence.

2. Lemma

The following lemma will be used in the proof of the theorem :

Lemma 1[2]—If $| c_k | \leqslant 1$, then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin (k + \frac{1}{2}) x}{2 \sin \frac{x}{2}} \right| dx \leqslant C (n + 1)$$

where C is a positive absolute constant.

3. Result

We prove the following result.

Theorem—Let (1.1) belongs to the class S. If $\lim_{n \to \infty} | a_{n+1} | \log n = 0$, then $\|f - g_n\| = o(1), n \to \infty$.

Proof : We have

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\Delta \left(\frac{a_k}{k}\right) + \Delta \left(\frac{a_{k+1}}{k+1}\right) + \ldots + \Delta \left(\frac{a_n}{n}\right) \right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1} \right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n'(x).$$

Now, making use of Abel's transformation and Lemma 1, we have

$$\int_0^{\pi} | f(x) - g_n(x) | \, dx \leqslant \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k \, D_k(x) \right| dx$$

(*equation continued on p.* 1103)

$$+ \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

$$= \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

$$= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx$$

$$+ \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

$$\leqslant \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx$$

$$+ \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

$$\leqslant C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx. \qquad \ldots(3.1)$$

Under the assumed hypothesis, $\Sigma (k+1) \Delta A_k$ converges and therefore the first term in (3.1) tends to zero as $n \to \infty$. Moreover, by Zygmund's Theorem[1] (p. 458).

$$\int_0^\pi \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx \leqslant \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

$$= \frac{|a_{n+1}|}{n+1} \int_{-\pi}^{\pi} | \widetilde{D}'_n(x) | \, dx$$

$$= C \, | a_{n+1} | \int_{-\pi}^{\pi} | \widetilde{D}_n(x) | \, dx$$

$$\sim | a_{n+1} | \log n \qquad \ldots(3.2)$$

since $\int_{-\pi}^{\pi} | \tilde{D}_n (x) | \, dx$ behaves like $\log n$.

The conclusion of the theorem now follows from (3.1) and (3.2).

Corollary—If (1.1) belongs to the class S and $\lim_{n\to\infty} | a_{n+1} | \log n = 0$, then $\|f - S_n\| = o(1)$, $n \to \infty$.

PROOF : We notice that

$$\int_{-\pi}^{\pi} | f(x) - S_n(x) | \, dx = \int_{-\pi}^{\pi} | f(x) - g_n(x) + g_n(x) - S_n(x) | \, dx$$

$$\leqslant \int_{-\pi}^{\pi} | f(x) - g_n(x) | \, dx + \int_{-\pi}^{\pi} | g_n(x) - S_n(x) | \, dx$$

$$\leqslant \int_{-\pi}^{\pi} | f(x) - g_n(x) | \, dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx.$$

Since $\lim_{n\to\infty} \int_{-\pi}^{\pi} | f(x) - g_n(x) | \, dx = 0$ by our theorem and $\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$ behaves like $| a_{n+1} | \log n$ by Zygmund's Theorem cited above, for large values of n, the conclusion of the corollary follows.

REFERENCES

1. N. K. Bari, *A treaties on trigonometric series*, Vol. II, Pergamon Press, London 1964.
2. G. A. Fomin, *Mat. Sbornik* 66 (107) (1964), 114–52.
3. B. Ram, *Proc. Am. Math. Soc.* 66 (1977), 158–260.
4. C. S. Rees and Č. V. Stanajević, *J. Math. Anal. Appl.* 43 (1973), 579–86.
5. S. Sidon, *J. London Math. Soc.* 14 (1939), 158–60.

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HYDRODYNAMIC STABILITY OF AN ANNULAR LIQUID JET HAVING A MANTLE SOLID AXIS USING THE ENERGY PRINCIPLE

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The hydrodynamic stability of an annular liquid jet having a regular or irregular mantle solid axis, subjected to capillary and inertia forces, is presented. An eigenvalue relation valid for all modes of perturbation is derived, using the energy principle. The characteristics of the model are identified analytically, confirmed numerically and interpreted physically. The model is stable to all non-axisymmetric modes but it is unstable only to axisymmetric sausage) modes whose wavelength is longer than the circumference of the liquid jet. However the maximum temporal amplification values prevailing of such model are far lower than that of the full liquid jet. The greater the radius of the cylindrical solid axis; the slower the corresponding growth rate values but the greater oscillation frequency values in the stability regions i. e., the thicker the solid mantle, the larger its stabilizing effect. The present results reduce to those of Rayleigh[12] if we impose that the radius of the cylindrical solid axis tends to zero.

1. INTRODUCTION

The experimental and theoretical hydrodynamic stability of a full liquid jet has been treated comprehensively since more than a century. This is not only from the academic viewpoint but also for its crucial applications in miscellaneous domains. Research in this field intensified when it became apparent that the physical properties of liquid jets play a fundamental role in a rapidly growing number of applications such as: the spinning of synthetic fibers, fuel atomization, spray drying, the production of controlled surfaces for heat and mass transfer in industrial and engineering processes and even the diagnosis of certain abnormalities of the human Urinary tract.

It was Plateau's observations[8] about the stability of a liquid jet (subjected to the curvature pressure) which led him to attribute the capillary instability to the surface tension. He was the first to obtain the critical wave length experimentally and theoretically using a naive approach. The decisive break through came with Rayleigh[10] who devised an elegant mathematical model for the breakup of liquids jets. Rayleigh[11] laid inexorable foundations for the theoretical treatment of such problems and developed the most important concept of the maximum mode of instability. By extending Rayleigh's theory, Weber[14] considered the capillary instability of a viscous liquid jet.

These and other various problems are summarized by Rayleigh[12] and, later on, also by Chandrasekhar[2] but with miscellaneous extensions.

The effect of non-linearities on the capillary instability of a hydrodynamical jet was examined by Yuen[15], Wang[13], Nayfeh[6], Nayfeh and Hassan[7] and a complete analysis was finally given by Kakutani *et al.*[3] using the derivative expansion approach developed by Kawahara[4].

The capillary instability of an annular liquid jet (a liquid jet having a vacuum or gas core jet) is also important to be investigated. The response of an incompressible-inviscid annular liquid jet subjecting to surface tension and the inertia forces (as the inertia force of the liquid is paramount over that of the gas jet) was given by Chandrasekhar[2] (p.540) for the axisymmetric mode only. Recently Kendall[5] have performed very interesting experiments with modern equipment for the capillary instability of an annular liquid jet for all modes of perturbations. Radwan[9] investigated analytically the capillary instability of that model in incisive account, and it is found that the theoretical results are in good agreement with Kendall's experimental results[5]. Indeed, Kendall[5] explained clearly and speculatively, in a large context, about the important applications of that model. Moreover he attracted the attention for the theoretical studies of the annular liquid jet in general.

The endeavours of the present work investigating the hydrodynamic stability of an annular liquid jet having a circular solid axis as a mantle subjecting to the capillary and inertia forces, by employing the energy principle.

The results of the present work reduce to those of Rayleigh[12] if we impose that the radius of the solid axis tends to zero.

2. Formulation and Eigenvalue Relation

We consider an incompressible-inviscid liquid jet of radius R coaxial with a solid axis of radius R_1 ($= qR$ where $0 < q < 1$) in equilibrium state. A cylindrical coordinates (r, φ z) system will be used with the z-axis coinciding with the axis of the coaxial (solid-liquid) cylinders. The influence of the vacuum medium surrounding the model is inevitable and neglected as long as the velocity of the perturbed liquid jet is not too large. The equilibrium density of the liquid ρ is assumed to be uniform. The gravitational effects are totally excluded.

To carry out the present analysis by employing the energy principle; we should find out the total kinetic energy E and the total potential energy V in order to write down the Lagrangian equation of motion. It may be noted that the Lagrangian function L is constructed as

$$L = E - V \qquad \ldots(1)$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial a^{*\sigma}}\right)-\frac{\partial L}{\partial a^*}=0 \qquad \ldots(2)$$

where a^* is the Lagrangian variable for the problem at hand and dot over a^* denotes the time derivative.

Let the equilibrium be disturbed then, in a cylindrical polar coordinates (r, φ, z) system. According to the linear theory, the perturbed (liquid-vacuum) interface will be described at any time by

$$r = R\,(1 + Q) \qquad \ldots(3)$$

with

$$Q = \sum_m \sum_k a_m\,(k) \exp i\,(k\,z + m\,\varphi). \qquad \ldots(3a)$$

Here Q is the elevation of the surface wave normalized with respect to R and measured from the equilibrium position. The coefficients $a_m\,(k)$ are functions of time and are much smaller than unity so that their higher orders can be neglected, based on the linear perturbation technique. Moreover $a_m(k)$ are complex and the condition that the elevation is real being

$$a_{-m}\,(-k) = a_k^*\,(k) \qquad \ldots(4)$$

where the asterisk* over $a_m\,(k)$ implies complex conjugate. m and k are the azimuthal and longitudinal wave-numbers respectively where m is integer while k may have all continuous (real -) values. Now since the motion is irrotational and the liquid is inviscid, the perturbed velocity vector $\mathbf{u}$ can be derived potentially viz.

$$\mathbf{u} = \text{grad}\,\phi. \qquad \ldots(5)$$

Using equation (5) with the equation of continuity for incompressible fluid, the potential velocity ϕ satisfies Laplace's equation

$$\nabla^2\,\phi = 0. \qquad \ldots(6)$$

Note that Euler's equation of motion has been used only implicitly in taking the irrotational flow as presistent. In view of the φ and z-dependence (cf. equation (3)); equation (6) reduces to an ordinary differential equation whose solution is given in terms of Bessel functions with argument of purely imaginary values. Under the present circumstances, the non-singular solution for ϕ must be

$$\phi\,(r;\,\varphi,\,z;\,t) = \sum_m \sum_k \left[b_m\,(k)\,I_m\,(kr) + c_m\,(k)\,K_m\,(kr)\right] \exp i\,(kz + m\,\varphi) \qquad \ldots(7)$$

where $b_m\,(k)$, $c_m\,(k)$ are unspecified functions of time and $I_m\,(kr)$, $K_m\,(kr)$ are the modified Bessel functions of order m of the first and second kind respectively.

Using eqns. (6) and (7); the total kinetic energy E can be expressed (on using Gauss theorem) as a surface integral in the form

$$E = \tfrac{1}{2} \rho \oint \phi \,(\text{grad}\, \phi .\, d\,\mathbf{S})$$

$$= \tfrac{1}{2} \rho \oint \phi \left(r \frac{\partial \phi}{\partial r} + r^{-1} \frac{\partial \phi}{\partial \varphi} \frac{dr}{d\varphi} + r \frac{\partial \phi}{\partial z} \frac{dr}{dz} \right) dz\, d\varphi \qquad \ldots(8)$$

where the surface elements $d\,S\,(= (r\, d\varphi\, dz,\, dr\, dz,\, r\, d\phi\, dr))$ in the orthogonal curvilinear cylindrical coordinates (r, φ, z) have been used. Since $\phi\,(r, \varphi, z; t)$ is of first order; eqn. (8) can be rewritten, up to second order, as

$$E = \tfrac{1}{2} \rho R \int dz \int_0^{2\pi} \phi \left(\frac{\partial \phi}{\partial r} \right)_{r=R} d\varphi. \qquad \ldots(9)$$

Substituting for ϕ from (7) into (9) and integrating with respect to φ; we get

$$E = \rho \pi R \int dz \sum_m \sum_{k,1} 1\, (b_m(k)\, I_m(kR) + c_m(k)\, K_m(kR))\, [b_{-m}(1)$$

$$I'_{-m}(\mathrm{R}) + c_{-m}(1)\; K'_{-m}(\mathrm{R})]\, \exp^2 i\,(kz + m\,\varphi) \qquad \ldots(10)$$

where the prime on Bessel functions denotes the derivative with respect to r and where k and l are different dimensional longitudinal wave-numbers.

The relation among the coefficients $a_m(k)$, $b_m(k)$ and $c_m(k)$ can be determined by imposing the boundary condition that the normal component of the velocity must be compatible with the deformed (liquid-vacuum) interface (3) at $r = R$ and that vanish at $r = R_1$. This yields

$$\frac{\partial \phi}{\partial r} = 0 \text{ at } r = R_1 = qR \qquad \ldots(11a)$$

$$= \frac{\partial r}{\partial t} \text{ at } r = R \qquad \ldots(11b)$$

from which

$$b_m(k) = -\, c_m(k)\, K'_m\, (qx)/I'_m\,(qx) \qquad \ldots(12a)$$

$$c_m(k) = x^{-1} R^2 a_m(k)\, I'_m\,(qx)\, [I'_m\,(qx)\, K'_m\,(x) - K'_m\,(qx)\, I'_m(x)]^{-1} \qquad \ldots(12b)$$

where $x\,(=kR)$ is the longitudinal non-dimensional wave-number.

Now, following Chandrasekhar[2] (p. 539), the potential energy V of a system arising from capillary forces is simply proportional to the total superficial area S; i. e.

$$V = T\,S \qquad \ldots(13)$$

where T is the surface tension coefficient. For the deformation given by (3), the total potential energy being

$$V = T \iint_{0}^{2\pi} \left(r^2 + \left(\frac{\partial r}{\partial \varphi}\right)^2 + r^2 \left(\frac{\partial r}{\partial z}\right)^2\right)^{1/2} dz\, d\varphi$$

$$= T\,R \int dz \int_{0}^{2\pi} (1 + Q)\,[1 + (1 + Q)^{-2} \left(\frac{\partial Q}{\partial \varphi}\right)^2 + R^2 \left(\frac{\partial Q}{\partial z}\right)^2]^{1/2}\, d\varphi. \qquad \ldots(14)$$

By a resort to equation (3) and that the liquid is incompressible (the volume is coserv-ed); equation (14) gives

$$V = 2\pi\, R\, T + \pi\, R\, T \int dz \sum_{m} \sum_{k,\,l} (-1 + l^2 R^2 + m^2)\, a_m(k)\, a^*_m\ (k)$$

$$\times \exp i\,(kz + m\varphi). \qquad \ldots(15)$$

Therefore, for a single mode (since there is no interference) given by

$$Q = a_m(k) \exp i\,(kz + m\varphi) + a_{-m}(-k) \exp(-i\,(kz + m\varphi))$$

$$= a_m(k) \exp i(kz + m\varphi) + a^*_m(k) \exp(-i\,(kz + m\varphi)) \qquad \ldots(16)$$

the total potential energy (per unit length in the z-direction) is given by

$$V = \pi R\, T\ \ [2 + (1 - m^2 - x^2)\ a_m(k)\ a^*_m(k)]. \qquad \ldots(17)$$

Similarly from eqns. (10) and (12) and for a mode given by (16); an expression for the total kinetic energy (per unit length in the z-direction) can written down.

By constructing Lagrangian function [cf. eqn. (1)], equation (2) gives following equation of motion

$$0 = a_m(k) = \frac{T}{\rho R^3}(1 - m^2 - x^2)\, a_m(k)$$

$$\frac{x\left(I'_m(x)\ K_m(qx) - I'_m(qx)\ K'_m(x)\right)}{\left(I_m(x)\ K'_m(qx) - I'_m\ (qx)\, K_m(x)\right)}. \qquad \ldots(18)$$

Now, as usual for stability problems; based on the linear perturbation technique, we assume that the time dependence is in the form exp ($\sigma\, t$); where σ the growth rate is

of the perturbation and if it is imaginary $\sigma = i\omega$ then $\omega/2\pi$ is the oscillation frequency. Henceforth equation (18) yields, at once, the eigenvalue relation

$$\sigma^2 = \frac{T}{\rho R^3}(1 - m^2 - x^2) F_m(x) \qquad \text{...(19)}$$

where

$$F_m(x) = \frac{x\left[I'_m(x) K'_m(qx) - I'_m(qx) K'_m(x) \right]}{\left[I_m(x) K'_m(qx) - I'_m(qx) K_m(x) \right]}. \qquad \text{...(19a)}$$

3. Discussion and Conclusion

Equation (19) is the desired eiegnvalue relation of an annular liquid jet having a circular solid axis as a mantle, subjecting to the liquid intertia force and endowed with surface tension at the (liquid-vacuum) interface. By means of that relation the characteristics of the present model can be determined: one can identify the instability regions (in particular their critical wave-numbers, maximum growth rate values and the corresponding wave-numbers) and those of stability as well.

The eigenvalue relation (19) relates the growth rate σ (or rather the oscillation frequency ω) with the entity $(T/\rho R^3)^{-1/2}$ as a unit of time, azimuthal wavenumber m, longitudinal non-dimensional wavenumber x, Wronskian like expression [cf. eqn (19a)], the geometric factor q (the radius of solid axis normalized with respect to that of the liquid cylinder) and the physical cylindrical functions appropriate to the present model.
As a limiting case as q tends to zero; equation (19) reduces to

$$\sigma^2 = \frac{T}{\rho R^3}(1 - m^2 - x^2) \; (x \; I'_m(x)/I_m(x)). \qquad \text{...(20)}$$

This is the classical dispersion relation of a full liquid cylinder, as was derived by Rayleigh; for its discussions we may refer to Chandrasekhar[2] (p. 537).

Other limiting case can be considered here as $q \lesssim 1$; the stability characteristics of such a case can be discussed as follows. Using a series development for the modified Bessel functions around the point qx, and neglecting terms of order $(1 - q)^2$, the eigenvalue relation (19) for $m = 0$ yields

$$\frac{\sigma^2}{T/\rho R^3} = x^2 (1 - x^2)(1 - q) \frac{I_1(x) K'_1(x) - I'_1(x) K_1(x)}{I_0(x) K_1(x) - K_0(x) I_1(x)}. \qquad \text{...(21)}$$

By the use of the well-known (Wronskian) relation

$$W(I_m(x) K_m(x)) = I_m(x) K'_m(x) - I'_m(x) K_m(x) = -x^{-1} \qquad \text{...(22)}$$

equation (21) gives

$$\frac{\sigma^2}{T/\rho R^3} = x^2 (1 - x)(1 - q). \qquad ...(23)$$

From equation (23) it is clear that σ is depleted when $q \lesssim 1$, so that the instability develops slower and slower, the thicker the solid cylinder is with respect to the liquid jet; as is intuitively clear. Taking the derivative of σ^2 we get

$$\frac{d\sigma}{dx} = (T/\rho\ R^3)^{1/2} (1 - q)^{1/2} (1 - x^2)^{1/2} (1 - 2x^2). \qquad ...(24)$$

From eqn. (24) we obtain directly that the maximum value of the growth rate at $x_m = 1/(2)^{1/2} = 0.707$; that agrees perfectly with the value determined numerically and is barely more than the x_m ($= 0.697$) for the full liquid jet.

By an appeal to the recurrence relations (cf. Abramowitz and Stegun[1]) of the modified Bessel functions

$$2\ I'_m (x) = I_{m-1} (x) + I_{m+1} (x) \qquad ...(25)$$

$$2K'_m (x) = -\ K_{m-1} (x) - K_{m+1} (x) \qquad ...(26)$$

and for each non-zero real value of x that $I_m (x)$ is always positive and monotonic increasing while $K_m (x)$ is monotonic decreasing but never negative; we can observe that $I'_m (x)$ is always positive while $K'_m (x)$ is always negative. According to the foregoing arguments; one can show that

$$F_m (x) > 0 \qquad ...(27)$$

for each non-zero real value of x, all q values and for all modes of perturbation; and that $F_m (x)$ is never change sign. By a resort to the inequality (27); eqn. (19) yields that $\sigma^2 < 0$ for all $m \neq 0$, but that $\sigma^2 > 0$ for $-1 < x < 1$ and $\sigma^2 \leqslant 0$ for $x \geqslant 1$ or $x \leqslant -1$ if $m = 0$. Therefore, the model is stable to all non-axisymmetric modes, but is unstable to axisymmetric (sausage) modes whose wave-length $\lambda = 2\pi/k$ is longer than the circumference $2\pi\ R$ of the liquid jet. Henceforth, for an annular liquid jet having a circular solid axis as a mantle we conclude that it is unstable only in the sausage mode in the domains $x \geqslant 1$ or $x \leqslant -1$; which is exactly the same for the full liquid jet subjected to the same forces as here. Indeed, this can be interpreted physically as follows. For such a model i. e. full liquid jet with or without solid axis; the potential energy due to surface tension (which is the only source of energy) can lower its value in the sausage mode $m = 0$ and hence give rise to kinetic energy, in particular there an enhancement in the very long wavelengths. That is also true even when the annular liquid jet having a vacuum or gas-core jet as a mantle (i. e. a hollow jet), see Chandrasekhar[2] (p. 540).

However for the present model, even if the solid axis is not regular we expect that the domains of instability are the same as if it is regular. One can make reasonable

estimates of the upper and lower limits of the growth rate as a function of x by considering the thinnest and thickest radius of the solid, say $q_1 R$ and $q_2 R$ respectively. Then one may expect that the dispersion curve for the model will be in between those of cylinders with values q_1 and q_2 respectively. Since the influence of q is not too large (see the numerical calculations Fig. 1), especially when $q < 0.7$ the upper and lower bounds will usually yield a good estimate if q_1 and q_2 do not differ too much. However if the internal solid has a periodic shape along its axis, there might turn up an enhanced effect in the dispersion relation for the corresponding wavelengths.

To identify more clearly the effect of q-values on the instability characteristics of the present model, the eigenvalue relation (19) with $m = 0$ (since there is no instability in the modes $m \neq 0$) and $x \geqslant 0$; has been used in the computer simulation for different values of q and for all (short and long) wave lengths. The numerical results are presented graphically, see Figs 1 and 2 for instability domains respectively. There are many

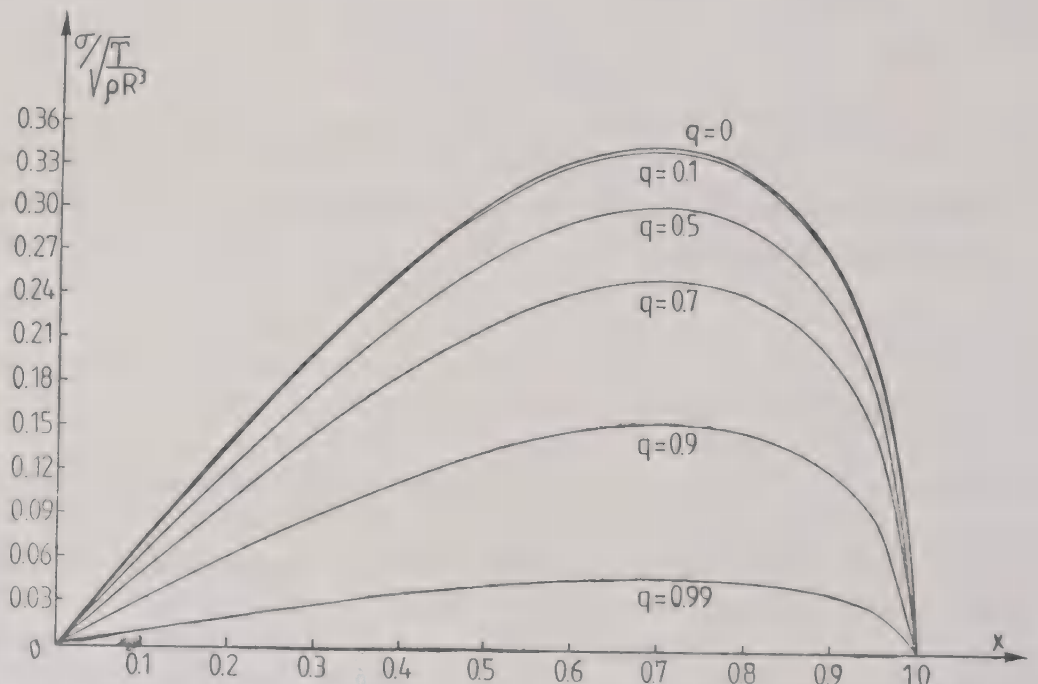

FIG. 1. The eigenvalue relation for the capillary instability of an annular liquid jet having a circular mantle solid axis (in the sausage mode $m = 0$ with $x < 1$). The abscissa measures the wavenumber in the unit $1/R$ and the ordinate the growth rate in the unit $(T/\rho R^3)^{1/2}$.

features in these figures. Although the domain of instability (mainly $0 \leqslant x < 1$ for $m = 0$) remains the same for all q-values; the area under the instability curves decreases with increasing q and rather proportionally over the whole domain of instability. Similarly we may prove for the stability curves (cf. figure 2).

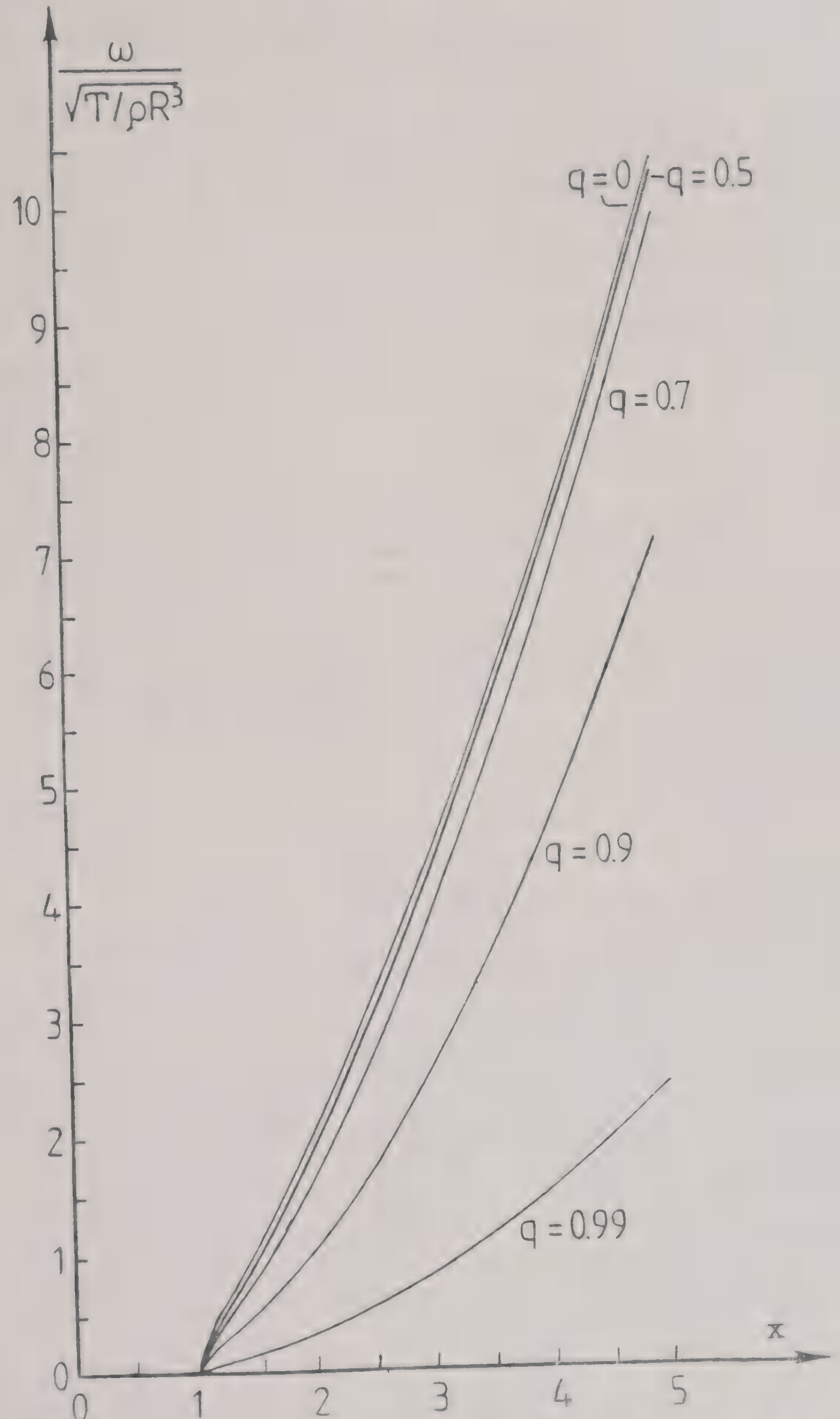

FIG. 2. The eigenvalue relation for the capillary stability of an annular liquid jet having a circular mantle solid axis (in the sausage mode $m = 0$ with $x \geqslant 1$). The abscissa measures the wavenumber in the unit $1/R$ and the ordinate the oscillation frequency in the unit $(T/\rho R^3)^{1/2}$.

Moreover from these numerical results we deduce that the larger x is, the smaller is the decrease in σ and ω with increase in q. However the effect is very small unless one considers very different wavenumbers. This result is in agreement with the very small shift in x_m from 0.697 to 0.707 for q increasing from 0 to unity. Particular attention is paid to the limiting cases $q \approx 0$ and $q \simeq 1$. A somewhat large number of digits were needed to determine the wavenumber corresponding to the maximum growth rate. In fact with four relevant digits for σ it was still not clear whether or not the x-values corresponding to the $\sigma_{\max}$ was the same for all q. Using greater precision will clarify that x-value corresponding to $\sigma_{\max}$ does shift from 0.697 to 0.707 as q increases from zero to unity. Therefore, one can state that the solid backbone has a stabilizing influence on the liquid: its influence is more stabilizing in case of larger values or q, i. e. the larger the ratio of the solid axis to the liquid jet. However the stabilizing effect is rather small: for a large q like 0.9, the σ-values decrease by a factor of about 2.3 and for a fairly extreme value like $q = 0.99$ they decrease by a factor of only 7.0. One can give a rough explanation for that weak stablizing influence as follows. As stated above the amount of energy sets free from the potential energy (the energy due to the curvature pressure) is the same for all q-values, but for a larger q (a relatively thicker solid cylinder), the amount of liquid requiring motion is smaller; thus the instability is not restrained very much in spite of the stabilizing effect of the solid which makes motion difficult.

For values of q exceeding say 0.95, we may use the approximation formula (23). However, it is worthwhile to mention here that when the fluid layer becomes rather thin several effects occur which may influence seriously the present analysis. One effect is that the solid surface may be uncovered, yielding an additional amount of energy as mentioned above. Other effect is the Maragoni effect which is due to fluctuations in the surface tension and which is relatively more important for thin layers.

References

1. M. Abramowitz and I. Stegum, *Handbook of Mathematical Function*. Dover Publ. New York, 1965, P. 376.
2. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Dover Publ., New York. 1981.
3. T. Kakutani, Y. Inoue and I. Kan, *J. Phys. Soc. Japan* 37 (1974), 529.
4. T. Kawahara, *J. Phys. Soc. Japan* 35 (1973) 1537.
5. J. M. Kendall, *Phys. Fluids* 29 (1986) 2086.
6. A. H. Nayfeh, *Phys. Fluids* 13 (1970) 841.
7. A. H. Nayfeh and S. Hassan, *J. Fluid Mech.* 48 (1971), 63.
8. M. T. Plateau, *Smithsonian Report*, (1863), 250.
9. A. E. Radwan, *Astrophys. Space Sci.* (1987), (to appear)
10. J. W. Rayleigh, *Proc. R. Soc.* 29 (1879), 71.
11. J. W. Rayleigh, *Scientific Papers*, Cambridge England (1899) p. 361.
12. J. W. Rayleigh, *The Theory of Sound*. Dover Publ., New York, 1945.
13. D. P. Wang, *J. Fluid Mech.* 34 (1968) 299.
14. C. Weber, *Z angew. Math. Mech.* 11 (1931), 136.
15. N. C. Yuen, *J. Fluid. Mech.* 33 (1968), 151.

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STRESS DISTRIBUTION AROUND TWO EQUAL CIRCULAR ELASTIC INCLUSIONS IN AN INFINITE PLATE UNDER THE ACTION OF AN ISOLATED FORCE APPLIED AT THE ORIGIN

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This paper contains the two-dimensional solutions of the problem in bipolar coordinates. The inclusions are elastic and perfectly bonded to an infinite elastic plate under the action of an isolated force. (i) X applied at the origin in the positive direction of x-axis, and (ii) Y applied at the origin in the positive direction of y-axis.

The stresses on the boundary, in each case, are found numerically and presented graphically. It is found that the boundary $\alpha_1 = 0.8$, the maximum values of the stresses $\widehat{\alpha\alpha}$ and $\widehat{\alpha\beta}$ are (i) $15.10X$ units attained at $\beta = 109.5°$ and $10.47X$ units attained at $\beta = 180$ respectively, in the first case; and (ii) $3.13Y$ units at $\beta = 0°$ and $25.64Y$ units at $\beta = 15°$ respectively, in the second case.

INTRODUCTIONS

The problem of stress distribution around elastic inclusions in an infinite plate has been studied by many authors. Saleme[1] has studied the stress distribution around a circular elastic inclusion in a semi infinite elastic plate under tension parallel to the straight edge. Mahata[2] has studied the stress distributions around two equal circular elastic inclusions in an infinite elastic plate under the action of (i) a centre of pressure radiating from a point, and (ii) a bi-axial or uni-axial tension.

In tbe present paper we have studied, in the absence of the body forces, the stress distribution around two equal circular elastic inclusions in an infinite elastic plate under the action of an isolated force (i) X applied at the origin in the positive direction of x-axis, in section 1, and (ii) Y applied at the origin in the position direction of y-axis, is secion 2.

In each section, we find two solutions, one of which is regular inside and the other is is regular outside the inclusions. Numerical calculations are performed and graphs of stresses are drawn.

The bipolar coordinates[3] are defined by

$$\alpha + i\beta = \log \frac{x + i(y + a)}{x + i(y - a)} \qquad \text{..(1.1)}$$

where a is a positive length.

$$\therefore\ x = \sin\beta/h,\ y = \sinh\alpha/h,\ ah = \cosh\alpha - \cos\beta. \qquad \text{...(1.2)}$$

Now, α = constant, represents a set of co-axial circles having the two poles $A\ (o, a)$ and $B\ (o, -a)$ of the above transformation, for limitimg points. The curves β = constant are a system of circles through A and B and intersecting the first set orthogonally. $\beta > 0$ on R. H. S. of y-axis and $\beta < 0$ on its L. H. S. while on the y-axis, $\beta = 0$, except on the segment AB where $\beta = \pm\pi$. At infinite, $\alpha = 0$, $\beta = 0$ and at A, B, $\alpha \to \infty$ and $\alpha \to -\infty$ respectively.

The stress function χ, in the absence of the body forces, satisfies the differential equation,

$$\left(\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\beta^2} + 1\right)(h\chi) = 0. \qquad \text{..(1.3)}$$

The stresses in terms of $h\chi$ are

$$a\widehat{\alpha\alpha} = \left[(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\beta^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cosh\alpha\right](h\chi) \qquad \text{...(1.4)}$$

$$a\widehat{\alpha\beta} = -(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha\,\partial\beta}(h\chi). \qquad \text{...(1.5)}$$

The components of the displacements are

$$Eu_\alpha = (1-\nu)\left(\frac{\partial}{\partial\alpha} - \frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\right)(h\chi) - \left(\frac{\partial}{\partial\beta} - \frac{\sinh}{\cosh\alpha - \cos\beta}\right)(hQ) \qquad \text{...(1.6)}$$

$$Eu_\beta = (1-\nu)\left(\frac{\partial}{\partial\beta} - \frac{\sin\beta}{\cosh\alpha - \cos\beta}\right)\cdot(h\chi) + \left(\frac{\partial}{\partial\alpha} - \frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\right)(hQ) \qquad \text{...(1.7)}$$

where hQ is the associated displacement function and satisfies the differential equation

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}(hQ) = \left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} - 1\right)(h\chi). \qquad \text{...(1.8)}$$

Method of Solution

Let the inclusions be defined by

$$\alpha = \pm\alpha_1,\ \alpha_1 > 0. \qquad \text{...(2.1)}$$

Section I

For an isolated force X applied at the origin in the positive direction of x-axis,

$$\chi_0 = -\frac{X}{2\pi}(y\theta - \nu x \log r) \quad ...(2.2)$$

where (x, y) and (r, θ) are the cartesian and polar coordinates of any point in the plate.

$$\therefore h\chi_0 = -\frac{X}{2\pi}\left[\sinh\alpha \tan^{-1}\left(\frac{\sinh\alpha}{\sin\beta}\right) - \tfrac{1}{2}\nu\sin\beta\log a - \tfrac{1}{2}\nu\sin\beta\log\left(\frac{\cosh\alpha + \cos\beta}{\cosh\alpha - \cos\beta}\right)\right]$$

$$= \frac{X}{2\pi}\left[-\frac{\pi}{2}\sinh\alpha + \sum_1^{\infty} K_n(\alpha)\sin n\beta\right] \quad ...(2.3)$$

where for $\alpha > 0$

$$\left.\begin{aligned} &K_1(\alpha) = -\left(\frac{\nu}{2} + 1\right)e^{-2\alpha} + 1 + \nu\log a \\ &\text{and for} \\ &n \geqslant 2,\ K_n(\alpha) = -e^{-n\alpha}\left[\{(-1)^n - 1\}\frac{\sinh\alpha}{n} - \frac{2\nu}{n^2 - 1}(n\sinh\alpha + \cosh\alpha)\right]. \end{aligned}\right\} \quad ...(2.4)$$

For the complete stress function we have to add to χ_0, a stress function χ_1 which will give no stress at infinity and satisfy eqn. (1.3). We assume,

$$h\chi_1 = \frac{X}{2\pi}\sum_1^{\infty}\phi_n(\alpha)\sin n\beta \quad ...(2.5)$$

where

$$\left.\begin{aligned} &\phi_1(\alpha) = A_1\cosh 2\alpha \\ \text{and}\quad &\phi_n(\alpha) = A_n\cosh(n+1)\alpha + B_n\cosh(n-1)\alpha,\ n \geqslant 2 \end{aligned}\right\}. \quad ...(2.6)$$

The associated displacement function hQ_1 corresponding to $h\chi_1$ is given by

$$hQ_1 = \frac{X}{2\pi}\sum_1^{\infty}\psi_n(\alpha)\cos n\beta \quad ...(2.7)$$

where

$$\left.\begin{aligned}\psi_1(\alpha) &= -2A_1 \sinh 2\alpha \\ \psi_n(\alpha) &= -2A_n\{\sinh(n+1)\alpha + B_n \sinh(n-1)\alpha,\}\ n \geqslant 2.\end{aligned}\right] \qquad \ldots(2.8)$$

and

The complete stress function is given by

$$h\chi = h\chi_0 + h\chi_1. \qquad \ldots(2.9)$$

The stresses $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ and displacements u_α, u_β corresponding to $h\chi$ are given by

$$\frac{a\widehat{\alpha\alpha}}{X/2\pi} = \sum_1^\infty [(1-n^2)\cosh\alpha\{\phi_n(\alpha) + K_n(\alpha)\} - \sinh\alpha\{\phi_n''(\alpha)$$

$$+ K_n'(\alpha)\} + \tfrac{1}{2}(n-1)(n-2)\{\phi_{n-1}(\alpha) + K_{n-1}(\alpha)\}$$

$$+ \tfrac{1}{2}(n+1)(n+2)\{\phi_{n+1}(\alpha) + K_{n+1}(\alpha)\}]\sin n\beta \qquad \ldots(2.10)$$

$$\frac{a\widehat{\alpha\beta}}{X/2\pi} = \phi_1'(\alpha) + K_1'(\alpha) + [-\cosh\alpha\{\phi_1'(\alpha)\} + K_1'(\alpha)\} + 2\{\phi_2'(\alpha)$$

$$+ K_2'(\alpha)\}]\cos n\beta$$

$$+ \sum_2^\infty \Big[-n\cosh\alpha\{\phi_n'(\alpha) + K_n'(\alpha)\}$$

$$+ \tfrac{1}{2}(n-1)\{\phi_{n-1}'(\alpha) + K_{n-1}'(\alpha)\} + \tfrac{1}{2}(n+1)\{\phi_{n+1}'(\alpha)$$

$$+ K_{n+1}'(\alpha)\}\Big]\cos n\beta \qquad \ldots(2.11)$$

$$\frac{Eh_{u\alpha}}{X/2\pi a} = (1+\nu)\left[\frac{\pi}{2}(1-\cosh\alpha\cos\beta) + \sum_1^\infty L_n(\alpha)\sin n\beta\right]$$

$$+ \sum_1^\infty [(1-\nu)\{\phi_n'(\alpha)\cosh\alpha - \tfrac{1}{2}\phi_{n-1}'(\alpha) - \tfrac{1}{2}\phi_{n+1}'(\alpha)$$

$$- \sinh\alpha\,\phi_n(\alpha)\} + n\,\psi_n(\alpha)\cosh\alpha - \tfrac{1}{2}(n-2)\psi_{n-1}(\alpha)$$

$$- \tfrac{1}{2}(n+2)\psi_{n+1}(\alpha)]\sin n\beta \qquad \ldots(2.12)$$

$$\frac{Ehu_\beta}{X/2\pi} = (1+\nu)\left[M_0(\alpha) - \frac{\pi}{2}\sinh\alpha\sin\beta + \sum_1^\infty M_n(\alpha)\cos n\beta\right]$$

$$- \{(1-\nu)\phi_1(\alpha) + \tfrac{1}{2}\psi_1'(\alpha)\} + \sum_1^\infty [(1-\nu)\{n\cosh\alpha$$

$$\phi_n(\alpha) - \tfrac{1}{2}(n-2)\phi_{n-1}(\alpha) - \tfrac{1}{2}(n+2)\phi_{n+1}(\alpha)\}$$

$$+ \cosh\alpha\,\psi_n'(\alpha) - \tfrac{1}{2}\psi_{n-1}'(\alpha)\,\tfrac{1}{2}\psi_{n+1}'(\alpha) - \sinh\alpha\,\psi_n(\alpha)]$$

$$\cos n\beta \qquad \text{...(2.13)}$$

where $L_1(\alpha) = 2e^{-\alpha}(\sinh 2\alpha - 1) - \frac{3\nu}{2}e^{-2\alpha}\sinh\alpha + \nu\sinh\alpha\log a$

for

$$n \geqslant 2,\ L_n(\alpha) = e^{-n\alpha}[\{(-1)^n - 1\}(\frac{1}{n} - \sinh\alpha) + (-1)^n\nu\sinh 2\alpha$$

$$+ \frac{e^{-n\alpha}}{n^2-1}[\{(-1)^n + 1\}\cosh\alpha(n\cosh\alpha + \sinh$$

$$\alpha) + 2\nu\sinh\alpha(n\sinh\alpha + \cosh\alpha)]$$

$$M_0(\alpha) = \nu(\log a - \tfrac{1}{2} - 2e^{-\alpha}\cosh\alpha) + (1 - e^{-2\alpha})$$

$$M_1(\alpha) = \nu(-\cosh\alpha\log a + 2e^{-\alpha} - \tfrac{3}{2}e^{-2\alpha}\cosh\alpha)$$

$$-\left(\frac{e^{-2\alpha}}{2},\sinh\alpha + e^{-\alpha}\sinh 2\alpha\right)$$

$$M_2(\alpha) = \frac{\nu}{2} + N_2(\alpha)$$

for $n \geqslant 2$,

$$N_n(\alpha) = \frac{2\nu}{n} + (-1)^n(1-\nu)\sinh 2\alpha\, e^{-n\alpha}$$

$$-[2\nu\cosh\alpha(n\cosh\alpha + \sinh\alpha)$$

$$+\{(-1)^n + 1\}\sinh\alpha(n\sinh\alpha + \cosh\alpha)]\frac{e^{-n\alpha}}{n^2-1}$$

and for $n > 3$, $N_n(\alpha) = M_n(\alpha)$.

Consideration, analogous to those introduced earlier, regarding the single valuedness of the displacement field, leads us to take the stress function within the inclussions, in the form,

$$h\bar{\chi} = \frac{X}{2\pi}\left[\bar{A}_0 \sinh\alpha + \sum_{1}^{\infty} \bar{\phi}_n(\alpha)\sin n\beta\right] \qquad \text{...(2.14)}$$

where

$$\left.\begin{aligned} \bar{\phi}_1(\alpha) &= A_1 e^{-2\alpha} \text{ for } \alpha > 0 \\ &= \bar{A}_1 e^{2\alpha} \text{ for } \alpha < 0 \\ \text{for } n \geqslant 2,\ \bar{\phi}_n &= \bar{A}_n e^{-(n+1)\alpha} + \bar{B}_n e^{-(n-1)\alpha} \text{ for } \alpha > 0 \\ &= \bar{A}_n e^{(n+1)\alpha} + \bar{B}_n e^{(n-1)\alpha}, \text{ for } \alpha < 0. \end{aligned}\right\} \qquad \text{...(2.15)}$$

and

The associated displacement function corresponding to $h\bar{\chi}$ is

$$h\bar{Q} = \frac{X}{2\pi}\sum_{1}^{\infty} \bar{\psi}_n(\alpha)\cos n\beta \qquad \text{...(2.16)}$$

where

and for

$$\left.\begin{aligned} \bar{\psi}_1(\alpha) &= 2\bar{\phi}_1(\alpha) \\ n \geqslant 2,\ \bar{\psi}_n(\alpha) &= 2\bar{\phi}_n(\alpha). \end{aligned}\right\} \qquad \text{...(2.17)}$$

The stresses within the inclusions are

$$\widehat{a\alpha\alpha} = \frac{X}{2\pi}\sum_{1}^{\infty}\Big[(1-n^2)\cosh\alpha\,\bar{\phi}_n(\alpha) + \tfrac{1}{2}(n-1)(n-2)\,\bar{\phi}_{n-1}(\alpha)$$

$$+ \tfrac{1}{2}(n+1)(n+2)\,\bar{\phi}_{n+1}(\alpha) - \sinh\alpha\,\bar{\phi}'_n(\alpha)\Big]\sin n\beta \qquad \text{...(2.18)}$$

$$\widehat{a\alpha\bar{\beta}} = \frac{X}{2\pi}\Big[\tfrac{1}{2}\bar{\varphi}'_1(\alpha) + \sum_{1}^{\infty}\Big\{\tfrac{1}{2}(n-1)\,\bar{\phi}'_{n-1}(\alpha) + \tfrac{1}{2}(n+1)\,\bar{\phi}'_{n+1}(\alpha)$$

$$- n\cosh\alpha\,\bar{\phi}'_n(\alpha)\Big\}\cos n\beta\Big]. \qquad \text{...(2.19)}$$

The displacements $\bar{u}_\alpha$ and $\bar{u}_\beta$ within the inclusions are for $\alpha > 0$,

$$\frac{\bar{E}h\bar{u}_\beta}{X/2\pi a} = -\bar{A}_0(1-\bar{\nu})\sinh\alpha\sin\beta + \tfrac{1}{2}\bar{\psi}'_1(\alpha) - (1-\bar{\nu})\,\bar{\phi}_1(\alpha)$$

(*equation continued on p.* 1121)

$$+ \sum_{1}^{\infty} [1 - \bar{\nu}) \{n \cosh \alpha\, \bar{\phi}_n (\alpha) - \tfrac{1}{2} (n-2)\, \bar{\phi}_{n-1} (\alpha)$$

$$- \tfrac{1}{2} (n+2)\, \bar{\phi}_{n+1} (\alpha)\}$$

$$- \cosh \alpha\, \bar{\psi}'_n (\alpha) + \tfrac{1}{2}\, \bar{\psi}'_{n-1} (\alpha) + \tfrac{1}{2}\, \bar{\psi}'_{n+1} (\alpha) + \sinh \alpha\, \bar{\psi}_n (\alpha)]$$

$$\times \cos n\beta \qquad \ldots(2.20)$$

$$\frac{\bar{E} h \bar{u}_\alpha}{X/2\pi a} = (1 - \bar{\nu})\, \bar{A}_0 (1 - \cosh \alpha \cos \beta) + \sum_{1}^{\infty} [(1 - \bar{\nu}) \{\cosh \alpha\, \bar{\phi}'_n (\alpha)$$

$$- \sinh \alpha\, \bar{\phi}_n (\alpha) - \tfrac{1}{2}\, \bar{\phi}'_{n-1} (\alpha) - \tfrac{1}{2}\, \bar{\phi}'_{n+1} (\alpha)\}$$

$$- n \cosh \alpha\, \bar{\psi}_n (\alpha) + \tfrac{1}{2} (n-2)\, \bar{\psi}_{n-1} (\alpha) + \tfrac{1}{2} (n+2)\, \bar{\psi}_{n+1}$$

$$(\alpha)] \sin n\beta.$$

Boundary Conditions

In case of perfect bond between the plate and the inclusions, we must have on $\alpha = \pm\, \alpha_1$,

$$\widehat{\alpha\alpha} = \widehat{\overline{\alpha\alpha}},\ \widehat{\alpha\beta} = \widehat{\overline{\alpha\beta}} \qquad \ldots(3.1)$$

$$u_\alpha = \bar{u}_\alpha,\ u_\beta = \bar{u}_\beta. \qquad \ldots(3.2)$$

By eqns (2.10), (2.11), (2.18), (2.19) and (3.1), we have

$$\therefore\ \tfrac{1}{2}\, \bar{\psi}_n (\alpha_1) = \bar{\phi}_n (\alpha_1) = \phi_n (\alpha_1) + K_n (\alpha_1) \qquad \ldots(3.3)$$

$$\times\, \tfrac{1}{2}\, \bar{\psi}'_n (\alpha_1) = \bar{\phi}'_n (\alpha_1) = \phi'_n (\alpha_1) + K'_n (\alpha_1), \qquad \ldots(3.4)$$

for $n = 1, 2, 3, \ldots$.

By eqns. (2.12), (2.13), (2.20), (2.21) and (3.2) we have

$$(1 + \nu)\, \frac{\pi}{2} = \frac{E}{\bar{E}} (1 - \bar{\nu})\, \bar{A}_0 \qquad \ldots(3.5)$$

$$A_1 = [(1 + \nu)\, M_0 (\alpha_1) + \frac{E}{\bar{E}} \{(1 - \bar{\nu})\, K_1 (\alpha_1) - K'_1 (\alpha_1)\}] / V'_2 (\alpha_1) \qquad \ldots(3.6)$$

$$3A_2 V_3(\alpha_1) + B_2\{3V_1(\alpha_1) - a' \sinh \alpha_1\} - A_1\{2V_2(\alpha_1)\cosh \alpha_1 + a' \sinh \alpha_1\} - \frac{E}{\bar{E}}[(1-\bar{\nu})\{K_1'(\alpha_1)\cosh \alpha_1 + \tfrac{1}{2}K_2'(\alpha_1) + K_1(\alpha_1)\sinh \alpha_1\} + 2K_1(\alpha_1)\cosh \alpha_1 - 3K_2(\alpha_1)] - (1+\nu) \times L_1(\alpha_1) = 0 \quad \ldots(3.7)$$

$$A_2 V_3'(\alpha_1) + B_2\{V_1'(\alpha_1) + a'\cosh \alpha_1\} - A_1\{V_2'(\alpha_1)\cosh \alpha_1 + 2V_1' \times (\alpha_1) - a'\cosh \alpha_1\} + \frac{E}{\bar{E}}[(1-\bar{\nu})\{K_1(\alpha_1)\cosh \alpha_1 - \tfrac{3}{2}K_2(\alpha_1)\} - 2K_1'(\alpha_1)\cosh \alpha_1 + K_2'(\alpha_1) + 2K_1(\alpha_1) \sinh(\alpha_1] - (1+\nu)M_1(\alpha_1) = 0 \quad \ldots(3.8)$$

and for $n \geqslant 2$, we have,

$$(n+2)V_{n+2}(\alpha_1)A_{n+1} + \{(n+2)V_n(\alpha_1) - a'\sinh n\alpha_1\}B_{n+1} - A_n[2nV_{n+1}(\alpha_1)\cosh \alpha_1 + a'\sinh n\alpha_1] - B_n[2n\cosh \alpha_1 V_{n-1}(\alpha_1) - a'\sinh n\alpha_1] + A_{n-1}[(n-2)V_n(\alpha_1) + a'\sinh n\alpha_1] + (n-2) \times V_{n-2}(\alpha_1)B_{n-1} + \frac{E}{\bar{E}}[(1-\bar{\nu})\{K_n'(\alpha_1)\cosh \alpha_1 - \tfrac{1}{2}K_{n+1}'(\alpha_1) - \tfrac{1}{2}K_{n+1}'(\alpha_1) - \sinh \alpha_1 K_n(\alpha_1)\} - 2n\cosh \alpha_1 K_n(\alpha_1) + (n+2) \times K_{n-1}(\alpha_1)(n+2)K_{n+1}(\alpha_1)] - (1+\nu)L_n(\alpha_1) = 0 \quad \ldots(3.9)$$

$$V_{n+1}'(\alpha_1)A_{n+1} + \{V_n'(\alpha_1) + a'\cosh n\alpha_1\}B_{n+1} - [2\{\cosh \alpha_1 V_{n+1}'(\alpha_1) - \sinh \alpha_1 V_{n+1}(\alpha_1)\} - a'\cosh n\alpha_1]A_n - [2\{\cosh \alpha_1 V_{n-1}'(\alpha_1) - \sinh \alpha_1 V_{n-1}'(\alpha_1)\} + a'\cosh n\alpha_1]B_n + [V_n(\alpha_1) - a'\cosh n\alpha_1] \times A_{n-1} + V_{n-2}'(\alpha_1)B_{n-1} + \frac{E}{\bar{E}}[(1-\bar{\nu})\{n\cosh \alpha_1 K_n(\alpha_1) - \tfrac{1}{2}(n-2)K_{n-1}(\alpha_1) - \tfrac{1}{2}(n+2)K_{n+1}(\alpha_1)\} - 2\cosh \alpha_1 K_n'(\alpha_1) + K_{n-1}'(\alpha) + K_{n+1}'(\alpha_1) + 2K_n(\alpha_1)\sinh \alpha_1] - (1+\nu)M_n(\alpha_1) = 0 \quad \ldots(3.10)$$

where

$$a' = 1 - \nu - \frac{E}{\bar{E}}(1 - \bar{\nu})$$

$$b = 1 + \nu - \frac{E}{\bar{E}}(1 - \bar{\nu})$$

$$V_n(\alpha) = -\frac{b}{2}\sinh n\alpha + \frac{E}{\bar{E}}\cosh n\alpha$$

$$V_n'(\alpha) = n\left(-\frac{b}{2}\cosh n\alpha + \frac{E}{\bar{E}}\sinh n\alpha\right).$$

Thus eqn. (3.5) gives $\bar{A}_0$; eqn. (3.6) gives A_1; eqns. (3.7) and (3.8) give A_2, B_2; and eqns. (3.9), (3,10) give A_n, B_n for $n = 3, 4, \ldots$: Finally the equations (2.10) and (2.11) give the stresses outside while eqns. (2.12) and (2.19) give those inside the inclusions.

Section II

For an isolated force[3] Y applied at the origin in the positive direction of y-axis,

$$\chi_0 = \frac{Y}{2\pi}(x\theta + \nu y \log r). \qquad \ldots(4.1)$$

Since the stresses and displacements corresponding to the stress function $\cosh\alpha - \cos\beta$ are zero, so any constant multiple of it is omitted from the stress function, and finally we have for $\alpha > 0$

$$h\chi_0 = \frac{Y}{2\pi}\left[\sinh\alpha\,(1 + \nu\log a) + \frac{\pi}{2}\sin\beta + \sum_1^{\infty} G_n(\alpha)\cos n\beta\right] \qquad \ldots(4.2)$$

where

$$G_1(\alpha) = \nu(1 - e^{-2\alpha}) - 1$$

and for

$$\left.\begin{aligned} n \geqslant 2,\ G_n(\alpha) &= \tfrac{1}{2}\{(-1)^n + 1\}\left[\frac{e^{-(n-1)\alpha}}{n-1} - \frac{e^{-(n+1)\alpha}}{n+1}\right] \\ &\quad + \frac{\nu}{n}\left[e^{-(n-1)\alpha} - e^{-(n+1)\alpha}\right]. \end{aligned}\right\} \qquad \ldots(4.3)$$

The associated displacement function hQ_0 is given by

$$\frac{hQ_0}{Y/2\pi} = -2\sum_1^{\infty} G_n(\alpha)\sin n\beta. \qquad \ldots(4.4)$$

Here we assume,

$$h\chi_1 = = \frac{Y}{2\pi}\left[B_0\,\alpha\,(\cosh\alpha - \cos\beta) + \sum_{1}^{\infty} \phi_n(\alpha)\cos n\boldsymbol{\beta}\right] \qquad \ldots(4.5)$$

where

$$\left.\begin{array}{ll} & \phi_1(\alpha) = C_1 \sinh 2\alpha \\ \text{and for} & \\ & n \geqslant 2,\ \phi_n(\alpha) = C_n \sinh(n+1)\,\alpha + D_n \sinh(n-1)\,\alpha \end{array}\right] \qquad \ldots(4.6)$$

$$\therefore\ hQ_1 = \frac{Y}{2\pi}\left[2B_0\,\boldsymbol{\beta}\cosh\alpha + \sum_{1}^{\infty} \psi_n(\alpha)\sin n\boldsymbol{\beta}\right] \qquad \ldots(4.7)$$

where

$$\left.\begin{array}{ll} & \psi_1(\alpha) = 2C_1 \cosh 2\alpha \\ \text{and for} & \\ & n \geqslant 2,\ \psi_n(\alpha) = 2\left[C_n \cosh(n+1)\,\alpha + D_n \cosh(n-1)\,\alpha\right] \end{array}\right]. \qquad \ldots(4.8)$$

The complete stress function is given by

$$h\chi = h\chi_0 + h\chi_1. \qquad \ldots(4.9)$$

The stresses $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ and the displacements u_α, u_β corresponding to $h\chi$ are given by

$$\begin{aligned} \frac{\widehat{a\alpha\alpha}}{Y/2\pi} = & -B_0 \sinh\alpha\,(\cosh\alpha - \cos\beta) + \phi_1(\alpha) + G_1(\alpha) \\ & + \sum_{1}^{\infty}\left[(1-n^2)\cosh\alpha\,\{\varphi_n(\alpha) + G_n(\alpha)\} + \tfrac{1}{2}(n-1)(n-2)\right. \\ & \times \{\phi_{n-1}(\alpha) + G_{n-1}(\alpha)\} + \tfrac{1}{2}(n+1)(n+2)\{\phi_{n+1}(\alpha) + G_{n+1}(\alpha)\} \\ & \left. - \sinh\alpha\,\{\phi_n'(\alpha) + G_n'(\alpha)\}\right]\cos n\beta \end{aligned} \qquad \ldots(4.10)$$

$$\begin{aligned} \frac{\widehat{a\alpha\beta}}{Y/2\pi} = & -B_0\cosh\alpha\sin\boldsymbol{\beta} + \frac{B_0}{2}\sin 2\beta + \sum_{1}^{\infty}\left[n\cosh\alpha\,\{\phi_n'(\alpha)\right. \\ & + G_n'(\alpha)\} - \tfrac{1}{2}(n-1)\,\{\phi_{n-1}'(\alpha) + G_{n-1}'(\alpha)\} - \tfrac{1}{2}(n+1) \\ & \left.\times \{\phi_{n+1}'(\alpha) + G_{n+1}'(\alpha)\}\right]\sin n\beta \end{aligned} \qquad \ldots(4.11)$$

$$\frac{aEhu_\alpha}{Y/2\pi} = B_0\left[\tfrac{1}{2}(1-\nu) - (1+\nu)\cosh^2\alpha + 2\nu\cosh\alpha\cos\beta + \tfrac{1}{2}(1-\nu)\right.$$

$$\left.\times\ \cos 2\beta + 2\beta\cosh\alpha\sin\beta\right] - \tfrac{1}{2}(1-\nu)\,\phi_1'(\alpha) + \psi_1(\alpha)$$

$$+ \frac{1+\nu}{2}G_1'(\alpha) + (1-\nu)(1+\nu\log a)(1-\cosh\alpha\cos$$

$$\beta) - \frac{\pi}{2}(1-\nu)\sinh\alpha\sin\beta + \sum_1^\infty ((1-\nu)[\cosh\alpha$$

$$\{\phi_n'(\alpha) + G_n'(\alpha)\} - \sinh\alpha\{\phi_n(\alpha) + G_n(\alpha)\} + \tfrac{1}{2}\{\phi_{n-1}'(\alpha) + G_{n\alpha 1}'(\alpha)\}$$

$$- \tfrac{1}{2}\{\phi_{n+1}'(\alpha) + G_{n+1}'(\alpha)\}] + n\cosh\alpha\{-\psi_n(\alpha) + 2G_n(\alpha)\}$$

$$+ (n+2)\{\tfrac{1}{2}\psi_{n+1}(\alpha) - G_{n+1}(\alpha)\} + (n-2)\{\tfrac{1}{2}\psi_{n-1} - G_{n-1}(\alpha)\})$$

$$\times\ \cos n\beta \qquad \ldots(4.12)$$

$$\frac{aEhu_\beta}{Y/2\pi} = -B_0\beta\sinh\alpha\cos\beta + (1-\nu)\left[-\frac{\pi}{2}(1+\cosh\alpha)\right.$$

$$+ \frac{\pi}{2}\cosh\alpha\cos\beta - \frac{\pi}{4}(\cosh\alpha - 1)\cos 2\beta - \sinh\alpha$$

$$\left.(1+\nu\log a)\sin\beta\right]$$

$$+ \sum_1^\infty ((1-\nu)[-n\cosh\alpha\{\phi_n(\alpha) + G_n(\alpha)\} + \tfrac{1}{2}(n-2)$$

$$\times \{\phi_{n-1}(\alpha) + G_{n-1}(\alpha)\} + \tfrac{1}{2}(n+2)\{\phi_{n+1}(\alpha) + G_{n+1}(\alpha)\}) + \cosh$$

$$\alpha\{\psi_n'(\alpha) - 2G_n'(\alpha)\} + \sinh\alpha\{-\psi_n(\alpha) + 2G_n(\alpha)\}$$

$$+ \{-\tfrac{1}{2}\psi_{n+1}'(\alpha) + G_{n-1}'(\alpha)\} + \{-\tfrac{1}{2}\psi_{n+1}'(\alpha) + G_{n-1}'(\alpha)\}$$

$$\sin n\beta. \qquad \ldots(4.13)$$

Within the inclusions, we take

$$h\bar{\chi} = \frac{Y}{2\pi}\left[\bar{B}_0\sin\beta + \sum_1^\infty \bar{\phi}_n(\alpha)\cos n\beta]\ \right] \qquad \ldots(4.14)$$

where

$$\left.\begin{aligned} \bar{\phi}_1(\alpha) &= C_1 e^{-2\alpha}, \text{ for } \alpha > 0 \\ &= \bar{C}_1 e^{2\alpha} \text{ for } \alpha < 0 \\ \text{and for} \\ n \geqslant 2,\ \bar{\phi}_n(\alpha) &= \bar{C}_n e^{-(n+1)\alpha} + \bar{D}_n e^{-(n-1)\alpha}, \text{ for } \alpha > 0 \\ &= \bar{C}_n e^{(n+1)\alpha} + \bar{D}_n e^{(n-1)\alpha}, \text{ for } \alpha < 0 \end{aligned}\right\} \quad \ldots(4.15)$$

$$\therefore h\bar{Q} = \frac{Y}{2\pi} \sum_{1}^{\infty} \bar{\psi}_n(\alpha) \sin n\beta \quad \ldots(4.16)$$

where

$$\left.\begin{aligned} \bar{\psi}_1(\alpha) &= -2\bar{\phi}_1(\alpha) \\ \text{and for} \\ n \geqslant 2,\ \bar{\psi}_n(\alpha) &= -2\bar{\phi}_n(\alpha) \end{aligned}\right]. \quad \ldots(4.17)$$

The stresses and displacements corresponding to $h\bar{\chi}$ are

$$\frac{a\widehat{\alpha\alpha}}{Y/2\pi} = \bar{\phi}_1(\alpha) + \sum_{1}^{\infty} [1 - n^2) \cosh\alpha\, \bar{\phi}_n(\alpha) + \tfrac{1}{2}(n-1)(n-2)\bar{\phi}_{n-1}(\alpha)$$

$$+ \tfrac{1}{2}(n+1)(n+2)\bar{\phi}_{n+1}(\alpha) - \sinh\alpha\bar{\phi}'_n(\alpha)] \text{ coc } n\beta. \quad \ldots(4.18)$$

$$\frac{a\widehat{\alpha\beta}}{Y/2\pi} = \sum_{1}^{\infty} (n\cosh\alpha\ \bar{\phi}'_n(\alpha) - \tfrac{1}{2}(n-)\bar{\phi}'_{n-1}(\alpha) - \tfrac{1}{2}(n+1)\bar{\phi}'_{n+1}$$

$$(\alpha)] \sin n\beta. \quad \ldots(4.19)$$

$$\frac{ah\bar{E}\bar{u}_\alpha}{Y/2\pi} = \frac{1+\bar{\nu}}{2}\bar{\phi}'_1(\alpha) + (1-\bar{\nu})[-\bar{B}_0 \sinh\alpha\sin\beta + \sum_{1}^{\infty}\{\cosh\alpha$$

$$\alpha\,\bar{\phi}'_n(\alpha) - \tfrac{1}{2}\bar{\phi}'_{n-1}(\alpha) - \tfrac{1}{2}\bar{\phi}'_{n+1}(\alpha) - \sinh\alpha\,\bar{\phi}_n(\alpha)\} \cos n\beta]$$

$$+ \sum_{1}^{\infty} [-\cosh\alpha\ \bar{\psi}_n(\alpha) + \tfrac{1}{2}(n-2)\ \bar{\psi}_{n-1}(\alpha)$$

$$+ \tfrac{1}{2}(n-2)\bar{\psi}_{n+1}(\alpha)] \cos n\beta \quad \ldots(4.20)$$

$$\frac{ah\bar{E}\bar{u}_\beta}{y/2\pi} = (1-\bar{\nu})[-\bar{B}_0(1-\cosh\alpha\cos\beta) + \sum_{1}^{\infty}\{-n\cosh\alpha\,\bar{\phi}_n(\alpha)$$

$$+ \tfrac{1}{2}(n-1)\bar{\phi}_{n-1}(\alpha) + \tfrac{1}{2}(n+2)\bar{\phi}_{n+1}(\alpha)\} \sin n\beta]$$

(*equation continued on p.* 1127)

$$+ \sum_{1}^{\infty} [\cosh \alpha \, \bar{\psi}'_n (\alpha) - \tfrac{1}{2} \, \bar{\psi}'_{n-1} (\alpha) - \tfrac{1}{2} \, \bar{\psi}'_{n+1} (\alpha) - \sinh \alpha$$

$$\bar{\psi}_n (\alpha)] \sin n\beta \cdot \qquad \text{...(4.21)}$$

Boundary Conditions

We have on the boundary $\alpha = \pm \alpha_1$,

$$\widehat{\alpha\alpha} = \widehat{\bar{\alpha\alpha}}; \; \widehat{\alpha\beta} = \widehat{\bar{\alpha\beta}} \qquad \text{...(5.1)}$$

$$u_\alpha = \bar{u}_\alpha, \, u_\beta = \bar{u}_\beta. \qquad \text{...(5.2)}$$

Therefore from (4.10), (4.11), (4.18), 4.19) and (5.1) we have,

$$B_0 = 0 \qquad \text{...(5.3)}$$

$$\left.\begin{aligned} \bar{\phi}_n (\alpha_1) &= \phi_n (\alpha_1) + G_n (\alpha_1) \\ \bar{\phi}'_n (\alpha_1) &= \phi'_n (\alpha) + G'_n (\alpha_1) \end{aligned}\right] \qquad \text{...(5.4)}$$

for $n = 1, 2, 3, \ldots$

From eqns. (4.12), (4.13), (4.20), (4.21) and (5.2), and using (5.4), we have

$$\bar{B}_0 = (1 - \nu) \, \pi/2 \, (1 - \bar{\nu}) \, \frac{E}{\bar{E}} \qquad \text{...(5.5)}$$

$$C_1 = - \left[\tfrac{1}{2} \, G'_1 (\alpha_1) + \frac{(1 - \nu)(1 + \nu \log a)}{1 + \nu - \dfrac{E}{\bar{E}} (1 + \bar{\nu})} \right] / \cos 2\alpha_1 \qquad \text{...(5.6)}$$

$$3C_2 (C_0 - a') \cosh 3\alpha_1 + D_2 (3C_0 - a') \cosh \alpha_1 + 2C_1 (a' - C_0 \cosh 2\alpha_1)$$

$$\times \cosh \alpha_1 + (2a' - C_0) \cosh \alpha_1 \, G'_1 (\alpha_1) - 2 \sinh \alpha_1 \, G_1 (\alpha_1)$$

$$- 3C_0 \, G_2 (\alpha_1) - a' \, G'_2 (\alpha_1) - 2 (1 - \nu)(1 + \nu \log a) \cosh \alpha_1 = 0 \qquad \text{...(5.7)}$$

$$3C_2 (a' - C_0) \sinh 3\alpha_1 + D_2 (3a' - C_0) \sinh \alpha_1 + 2C_1 [(C_0 - a') \cosh$$

$$\alpha_1 \sinh 2\alpha_1 + C_0 \sinh \alpha_1] + 3G_2 (\alpha_1) (a' - \frac{C_0}{4})$$

$$- \tfrac{1}{2} (1 + \frac{C_0}{2}) \, G''_2 (\alpha_1) - 2a' \cosh \alpha_1 \, G_1 (\alpha_1) - C_0 \sinh \alpha_1 \, G'_1$$

(equation continued on p. 1128)

$$(\alpha_1) + C_0 \cosh \alpha_1\, G_1''(\alpha_1) - 2(1-\nu)(1+\nu \log a) \sinh \alpha_1 = 0 \quad \text{.. (5.8)}$$

and for $n \geqslant 2$, we have,

$$R'_{n+2}(\alpha_1)\, C_{n+1} + D_{n+1}\left[\frac{n+2}{n} R'_n(\alpha_1) + a' \operatorname{cosn} n\alpha_1\right]$$

$$- C_n\left[\frac{2n}{n+1} \cosh \alpha_1\, R'_{n+1}(\alpha_1) + a' \cosh n\alpha_1\right]$$

$$- D_n\left[\frac{2n}{n-1} \cosh \alpha_1\, R'_{n-1}(\alpha_1) + a' \cosh n\alpha_1\right]$$

$$+ C_{n-1}\left[\frac{n-2}{n} R'_n(\alpha_1) - a' \cosh n\alpha_1\right] + D_{n-1}\, R'_{n-2}(\alpha_1)$$

$$+ a'\left[\cosh \alpha_1\, G'_n(\alpha_1) - \sinh \alpha_1.\, G_n(\alpha_1) - \tfrac{1}{2} G'_{n-1}(\alpha_1)\right.$$

$$\left. - \tfrac{1}{2} G'_{n+1}(\alpha_1)\right] + C_0\left[n \cosh \alpha_1\, G_n(\alpha_1) - \frac{n+2}{2} G_{n+1}(\alpha_1)\right.$$

$$\left. - \frac{n-2}{2} G_{n-1}(\alpha_1)\right] = 0 \quad \text{...(5.9)}$$

$$(n+2) R_{n+2}(\alpha_1)\, C_{n+1} + D_{n+1}[nR_n(\alpha_1) - a' \sinh n\alpha_1] - C_n[a' \sinh n\alpha_1$$

$$+ 2(n+1) \cosh \alpha_1\, R_{n+1}(\alpha_1) - \frac{2 \sinh \alpha_1}{n+1} R'_{n+1}(\alpha_1)]$$

$$- D_n[2(n-1) \cosh \alpha_1\, R_{n-1}(\alpha_1) - \frac{2 \sinh \alpha_1}{n-1} R'_{n-1}(\alpha_1)$$

$$- a' \sinh n\alpha_1] + C_{n-1}[nR_n(\alpha_1) + a' \sinh n\alpha_1]$$

$$+ (n-2) R_{n-2}(\alpha_1)\, D_{n-1} + a'[n \cosh \alpha_1\, G_n(\alpha_1) - \frac{n-2}{2}$$

$$\times G_{n-1}(\alpha_1) - \frac{n+2}{2} G_{n+1}(\alpha_1)] + C_0[\cosh \alpha_1\, G'_n(\alpha_1)$$

$$- \sinh \alpha_1\, G_n(\alpha_1) - \tfrac{1}{2} G'_{n-1}(\alpha) - \tfrac{1}{2} G'_{n+1}(\alpha_1))] = 0 \quad \text{...(5.10)}$$

where $C_0 = 2(1 - E/\bar{E})$

$$R_n(\alpha) = \frac{b}{2} \sinh n\alpha + \frac{E}{\bar{E}} \cosh n\alpha$$

$$R'_n(\alpha) = n\left(\frac{b}{2} \cosh n\alpha + \frac{E}{\bar{E}} \sinh n\alpha\right).$$

Thus (5.3) gives B_0, the equation (5.5) gives $\bar{B}_0$, the equation (5.6) gives C_1, the equations (5.7), (5.8) give C_2, D_2; and equations (5.9), (5.10) give C_n, D_n for $n = 3, 4, \ldots$ Finally, the equations (4.10), (4.11) give the stresses outside and eqns. (4.18) and (4.19) give those inside the inclusions.

Numerical Calculation

Numerical Calculations are carried out with $a = 1$, $\nu = \nu = \frac{1}{3}$, $E/\bar{E}$ $\frac{1}{2}$, $\alpha_1 = 0.8$. The circumferential stresses are found Table 1 and are shown graphically for both the cases against β (Fig. 1, 2).

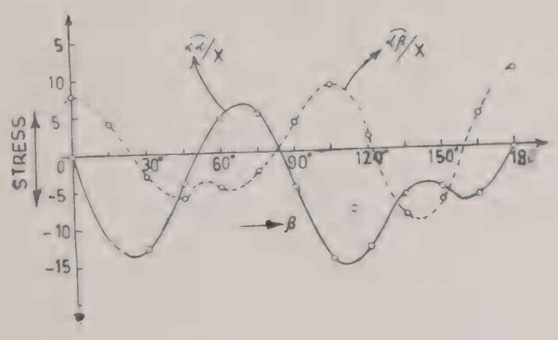

Fig. 1. Stresses on the boundary $\alpha_1 = 0.8$ under the force X along the x-axis.

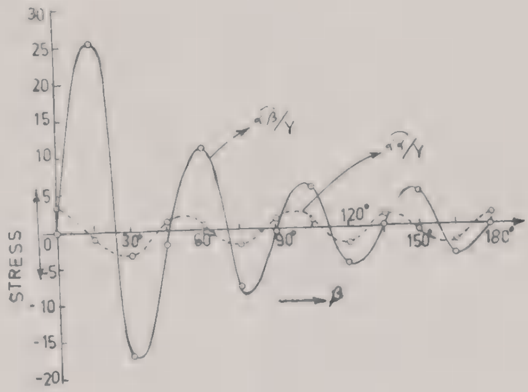

Fig. 2. Stresses on the boundary $\alpha_1 = 0.8$ under the force Y along the y-axis.

Table I

Max. value of the stress	For isolated force X parallel to x-axis	For isolated force Y-parallel to y-axis
$\widehat{\alpha\alpha}$	$15.10X$ at $\beta = 109.5°$	$3.13Y$ at $\beta = 0°$
$\widehat{\alpha\beta}$	$10.47X$ at $\beta = 180°$	$25.64Y$ at $\beta = 15°$

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References

1- E. M. Saleme, *J. Appl. Mech.* 25 (1958), 129.
2. S. Mahata, *Indian J. pure appl. Math.* 18, 74-82; *Bull. Cal. Math. Soc.* 79 (1987), 158-69.
3. G. B. Jeffery, *Phil. Trans. Roy. Soc.* (London).

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THREE DIMENSIONAL CONVECTIVE FLOW AND HEAT TRANSFER IN A POROUS MEDIUM

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The effect of periodic variation of suction velocity on free convection flow and heat transfer through a porous medium is investigated. The problem becomes three dimensional due to variation of suction velocity in transverse direction on the wall. A series expansion method is used to get the solution of the governing equations and the expressions for velocity and temperature fields are obtained. The skin friction and the rate of heat transfer at the wall are analysed in detail.

INTRODUCTION

The problem of laminar flow control has gained considerable importance in the field of Aeronautical Engineering in view of its application to reduce drag and hence the vehicle power requirement by a substantial amount. The development on this subject has been compiled by Lachmann[1]. Theoretical and experimental investigations have shown that the transition from laminar to turbulent flow which causes the drag coefficient to increase, may be prevented by the suction of fluid and heat transfer from boundary layer to the wall. Gersten and Gross[2] have studied the effect of transverse sinusoidal suction velocity on flow and heat transfer along an infinite vertical porous wall. The flow and heat transfer through a porous medium, however, has not received much attention despite its application in many branches of engineering. In this note, the effect of suction velocity variation on free convective flow and heat transfer in a porous medium is investigated.

ANALYSIS

We consider free convection flow of a viscous incompressible fluid through a porous medium bounded by an infinite vertical porous wall. A coordinate system with the wall lying on $x - z$ plane and y-axis perpendicular to it and directed into the fluid is introduced. The suction velocity distribution is taken in the form :

$$v_w(z) = v_0 \left[1 + \epsilon \cos \frac{\pi z}{L} \right] \qquad ...(1)$$

which consists of a basic steady distribution $V_0 < 0$ with a superimposed weak transversally varying distribution $\epsilon V_0 \cos \pi z/L$ of wave length L. Since the wall is infinite in x-direction, hence all physical quantities will be independent of x, however, the flow remains three dimensional due to variation of suction velocity. Let us denote velocity components u, v, w in x, y, z directions respectively and temperature by T. The governing equations are

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(2)$$

$$v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = g\beta (T - T_\infty) + \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\nu u}{K} \qquad ...(3)$$

$$v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\nu v}{K} \qquad ...(4)$$

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\nu w}{K} \qquad ...(5)$$

$$v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \qquad ...(6)$$

In eqn. (3), variation in density is taken into account only in the derivation of the buoyancy force while other density variations are neglected within the frame-work of constant property fluid. The last terms in eqns. (3), (4) and (5) account for the pressure drop across the porous material. It is worthwhile to mention that the basic flow in the medium is entirely due to buoyancy force, caused by temperature difference between the wall and the medium.

The boundary conditions of the problem are :

$$\left. \begin{array}{l} y = 0 : u = 0, v = v_w(z), w = 0, T = T_w \\ y \to \infty : u = 0, w = 0, p = p_\infty, T = T_\infty. \end{array} \right\} \qquad ...(7)$$

When the amplitude ϵ of oscillations in suction velocity is small, we assume the main flow velocity u in the following form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + .. \qquad ...(8)$$

The similar equations hold for other variables v, w, p and $\theta = \dfrac{T - T_\infty}{T_w - T_\infty}$. When $\epsilon = 0$, the problem reduces to two-dimensional free convection flow in an infinite porous medium with constant suction velocity at the wall. In this case eqns. (2) to (6) reduce to

$$\frac{\partial v_0}{\partial y} = 0 \qquad ...(9)$$

$$v_0 \frac{du_0}{dy} = g\beta (T_w - T_\infty)\,\theta_0 + \nu \frac{d^2u_0}{dy^2} - \frac{\nu u_0}{K} \qquad \text{...(10)}$$

$$v_0 \frac{d\theta_0}{dy} = \alpha \frac{d^2\theta_0}{dy^2}\,. \qquad \text{...(11)}$$

The solution of these equations is

$$u_0 = \frac{G\nu}{Lp_1} (e^{-m\bar{y}} - e^{-PR\bar{y}}),\ v_0 = v_0,\ w_0 = 0 \qquad \text{...(12)}$$

$$\theta_0 = e^{-PR\bar{y}},\ p_0 = p_\infty \qquad \text{...(13)}$$

where

$$\bar{Y} = y/L,\ P = \nu/\alpha,\ G = \frac{g\beta (T_w - T_\infty)\,L^3}{\nu^2},\ P_1 = R^2 (P^2 - P) - K_1$$

$$R = -\frac{LV_0}{\nu},\ K_1 = \frac{L^2}{K},\ m = \frac{R}{2} + (R^2/4 + K_1)^{1/2}.$$

When $\epsilon \neq 0$, the series expansion (8) is substituted in equations (2) to (6) and like powers of ϵ are equated to get the perturbation equations of various order in ϵ. For small values of ϵ, it is sufficient to consider the perturbation equations only of $o(\epsilon)$, which are

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \qquad \text{...(14)}$$

$$v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} = \frac{G\nu^2}{L^3}\,\theta_1 + \nu \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) - \frac{\nu}{K} u_1 \qquad \text{...(15)}$$

$$v_0 \frac{\partial v_1}{\partial y} = -\frac{1}{\rho} \frac{\partial P_1}{\partial y} + \nu \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) - \frac{\nu}{K} v_1 \qquad \text{...(16)}$$

$$v_0 \frac{\partial w_1}{\partial y} = -\frac{1}{\rho} \frac{\partial p_1}{\partial z} + \nu \left(\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} \right) - \frac{\nu}{K} w_1 \qquad \text{...(17)}$$

$$v_0 \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \theta_0}{\partial y} = \alpha \left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} \right) \qquad \text{...(18)}$$

with boundary conditions

$$\left.\begin{aligned} &y = 0 : u_1 = 0,\ v_1 = V_0 \cos \pi \frac{z}{L},\ w_1 = 0,\ \theta_1 = 0 \\ &y \to \infty : u_1 = 0,\ w_1 = 0,\ p_1 = 0,\ \theta_1 = 0. \end{aligned}\right\} \qquad \text{...(19)}$$

This is a set of linear partial differential equations which describe the three-dimensional cross flow.

First of all we shall consider the eqns. (14), (16) and (17). The solutions for $v_1(y, z)$, $w_1(y, z)$ and $p_1(y, z)$ are independent of main flow component u_1 and temperature field θ_1. We now assume v_1, w_1 and p_1 as

$$v_1(\bar{y}, \bar{z}) = V_0 \, \pi \, v_{11}(\bar{y}) \cos \pi\bar{z} \quad \text{...(20)}$$

$$w_1(\bar{y}, \bar{z}) = - v_0 \, v_{11}^1(\bar{y}) \sin \pi\bar{z} \quad \text{...(21)}$$

$$p_1(\bar{y}, \bar{z}) = \rho \, V_0^2 \, p_{11}(\bar{y}) \cos \pi\bar{z} \quad \text{...(22)}$$

where $\bar{z} = z/L$ and prime denotes differentiation with respect to $\bar{y}$.

Equations (20) and (21) have been chosen so that the continuity equation (14) is satisfied. Substituting these expressions into (16) and (17) and applying the corresponding transformed boundary conditions, we get the values of v_1, w_1 and p_1 as :

$$v_1(\bar{y}, \bar{z}) = \frac{V_0}{\pi - \lambda} (\pi e^{-\lambda\bar{y}} - \lambda e^{-\pi\bar{y}}) \cos \pi\bar{z} \quad \text{...(23)}$$

$$w_1(\bar{y}, \bar{z}) = \frac{\lambda V_0}{\pi - \lambda} (e^{\lambda\bar{y}} - e^{-\pi\bar{y}}) \sin \pi\bar{z} \quad \text{...(24)}$$

$$p_1(\bar{y}, \bar{z}) = N_2 \, \rho \, V_0^2 \left(\frac{\lambda}{\pi - \lambda}\right) e^{-\pi\bar{y}} \cos \pi\bar{z} \quad \text{...(25)}$$

where

$$N_2 = 1 + \frac{K_1}{R\pi} \text{ and } \lambda = \frac{R}{2} (R^2/4 + \pi^2 K_1)^{1/2}.$$

Assuming u_1 and θ_1 of the form

$$u_1(\bar{y}, \bar{z}) = u_{11}(\bar{y}) \cos \pi\bar{z} \quad \text{...(26)}$$

$$\theta_1(\bar{y}, \bar{z}) = \theta_{11}(\bar{y}) \cos \pi\bar{z} \quad \text{...(27)}$$

and substituting in eqns. (15) and (18), we get

$$u_1(\bar{y}, \bar{z}) = \frac{\nu}{L} \frac{G}{P_1} \frac{R}{\pi - \lambda} \Big[- \left(\frac{\pi}{2\lambda N_4} - \frac{\lambda}{\pi N_3} - A_1 + A_2 + A_3 \right) e^{-\lambda\bar{y}}$$

$$+ \frac{\pi}{2\lambda N_4} e^{-(\lambda+m)\bar{y}} - \frac{\lambda}{\pi N_3} e^{-(\pi+m)\bar{y}} - A_1 e^{-(\lambda+PR)\bar{y}}$$

$$+ A_2 e^{-(\pi+PR)\bar{y}} + A_3 e^{-\bar{\lambda}\bar{y}} \Big] \cos \pi\bar{z} \quad \text{...(28)}$$

$$\theta_1(\bar{y}, \bar{z}) = \frac{P^2 R}{\pi - \lambda} \left[\frac{\pi}{P_2} e^{-(\lambda+PR)\bar{y}} - \frac{\lambda}{\pi P} e^{-(\pi+PR)\bar{y}} - \frac{A_3}{P_3} e^{-\bar{\lambda}\bar{y}} \right] \cos \pi\bar{z} \quad \text{...(29)}$$

where

$$P_2 = \lambda(1 + P) + K_1/R,\ P_3 = \frac{P_1 P^2}{R\bar{\lambda}(P-1) - K_1}$$

$$N_3 = \frac{1}{m}(R^2 + 4K_1)^{1/2}$$

$$N_4 = 1 + K_1/2m\lambda,\ A_1 = \frac{\pi(1 + PP_1/RP_2)}{PR + 2\lambda - R}$$

$$A_2 = \frac{\lambda P(R + P_1/\pi)}{P^2 R^2 + 2\pi PR - R^2 P - \pi R - K_1}$$

$$A_3 = P_3(\pi/P_2 - \lambda/\pi P),\ \bar{\lambda} = \frac{PR}{2} + (\pi^2 + P^2R^2/4)^{1/2}.$$

Results and Discussions

We now discuss the important flow characteristics of the problem. The expression for shear stress along the direction of x is obtained as

$$C_{fx} = \frac{\tau_x}{\rho\nu^2/L^2} = \frac{\mu}{\rho}\frac{\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\nu^2/L^2}$$

$$= \frac{G}{P_1}(PR - m) + \epsilon GF_1(P, R)\cos\pi\bar{z} \qquad \ldots(30)$$

where

$$F_1(P, R) = \frac{R}{P_1(\pi - \lambda)}\left[-\frac{\pi m}{2\lambda N_4} + \frac{\lambda}{\pi N_3}(\pi + m - \lambda) + A_1 PR + A_2(\lambda - PR - \pi) + A_3(\lambda - \bar{\lambda})\right] \qquad \ldots(31)$$

The skin friction factor $F_1(P, R)$ in the limiting cases, $R \to 0$ and $R \to \infty$, tends to zero. Equation (31) is numerically evaluated for different values of R and permeability parameter K_1 and the results are shown in Fig. 1. From Fig. 1, it is obvious that F_1 tends to its limiting values as $R \to 0$ and $R \to \infty$ for each K_1. It is also seen that F_1 increases with R until it attains a maximum value, after which it decreases and approaches to zero. It has also been observed that, as the value of K_1 is increased, the skin friction factor F_1 decreases significantly.

The heat flux at the wall in terms of Nusselt number Nu is given by

$$Nu = \frac{-q_w}{\rho V_0 c_p(T_w - T_\infty)} = \frac{k}{\rho V_0 c_p}\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

$$= 1 + \epsilon[1 - F_2(P, R)]\cos\pi\bar{z} \qquad \ldots(32)$$

where

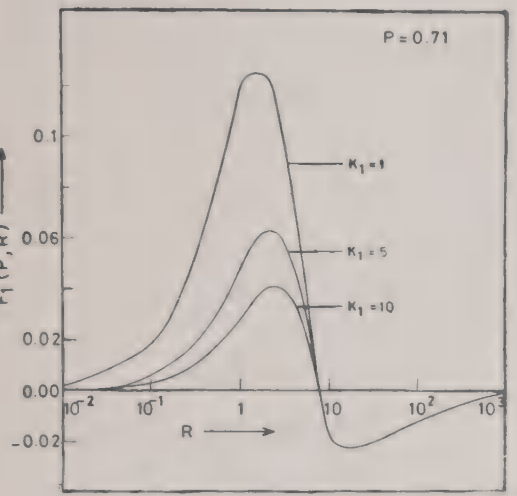

FIG. 1. Variation of skin friction factor F_1 with R for different values of permeability parameter K_1.

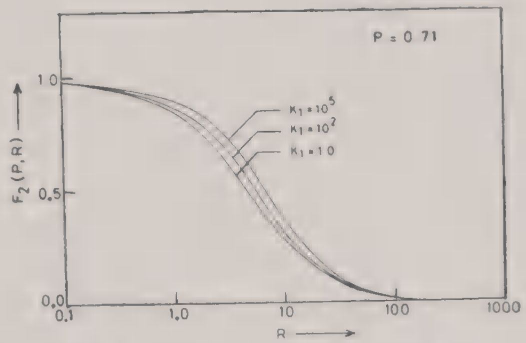

FIG. 2. Variation of heat transfer factor F_2 with R for different values of permeability parameter K_1.

$$F_2 (P, R) = \frac{1}{\lambda - \pi} \left[\lambda \left(\frac{\lambda}{\pi} - \frac{\pi P}{P_2} \right) + \frac{\pi P}{P_2} (\lambda + PR) - \frac{\lambda PR}{\pi} - \pi \right]. \quad \text{...(33)}$$

It is found that F_2 takes constant values for limiting values of R. When $R \to 0$, $F_2 \to 1$ which is obvious because there is no oscillatory flow. However, when $R \to \infty$, $F_2 \to 0$ which shows that heat transfer approaches to quasi-steady value. These results are shown in Fig. 2. It is interesting to note that $F_2 (P, R)$ increases with permeability parameter K_1.

ACKNOWLEDGEMENT

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REFERENCES

1. G. V. Lachmann, *Boundary Layer and Flow Control. Its principles and Applications.* Vols I and II, Pergamon Press, 1961.
2. K. Gersten and J. F. Gross, *ZAMP* 25 (1974), 399–408.

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MHD SWIRLING JET WHICH ORIGINATES FROM A CIRCULAR SLIT

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The effects of an axial magnetic field, which varies inversely as the radius vector, on the velocity distributions in a swirling jet of a viscous incompressible electrically conducting fluid are studied. It is noted that the radial as well as tangential velocity decrease near the slit of the jet with the increase in the value of the magnetic interaction parameter. These decelerated fluid particles move in the positive axial direction and the points where they exactly balance the motion of the incoming fluid have been calculated for different values of the magnetic interaction parameter.

NOMENCLATURE

$\vec{B}$ = magnetic field vector

B_0 = axial magnetic field at a unit radial distance

F_{jl} = function introduced in eqn. (3.2)

G_{jl} = function introduced in eqn. (3.3)

J_0 = linear momentum flux at a large distance from the slit

L_0 = initial angular momentum flux

m = $\dfrac{\sigma_e B_0^2}{\rho}$ (magnetic parameter)

Q = volume flux in the radial direction

r = radial coordinate

u = radial velocity component

v = axial velocity component

w = circumferential velocity component

z = axial coordinate

Greek Letters

$\alpha = \left(\dfrac{3J_0}{4\pi \rho \sqrt{\nu}} \right)^{1/3}$

$\beta = \left(\dfrac{L_0}{J_0} \right)^{1/2}$

ξ = non-dimensional variable $\left(= \dfrac{\alpha z}{r\sqrt{\nu}} \right)$

μ = coefficient of dynamic viscosity

ρ = density

ν = kinematic viscosity

σ_e = electrical conductivity

ψ = Stokes' stream function.

1. Introduction

The flow of a laminar jet issuing from a slit (plane free jet) and a jet issuing from a circular orifice (circular free jet) without swirl, of an electrically non-conducting, incompressible viscous fluid were investigated by Schlichting[1], Bickley[2] and the corresponding MHD flow has been studied by a number of workers including Peskin[3], Smith and Cambel[4], Pozzi and Bianchini[5], Bansal[6], Gupta[7] and Mishra and Bansal[8].

In swirling flows, besides axial and radial velocity components, the tangential component of velocity should also be considered. It is both the linear momentum and the angular momentum of developing swirling flow which play an important role in determining its ultimate form. The case of a laminar swirling jet of a non-conducting, viscous, incompressible fluid was considered by Loitsianski[9], Görtler[10], Steiger and Bloom[11] and solutions were obtained using the assumption of similarity of the velocity profiles. Similarity in turbulent swirling wakes and jets have been studied by Reynolds[12] and Chervinsky[13]. Chervinsky noted that for the turbulent axisymmetrical swirling jets, the asymptotic solution is valid only in regions where the axial pressure gradient is negligibly small.

Recently, Mishra and Bansal[14] studied the effect of circular magnetic field on the decay of a weak swirl in the axially-symmetrical circular free jet of an electrically conducting, viscous fluid and found that the swirl velocity increases due to the presence of the magnetic field, though its decay is inversely proportional to the distance square along the axis of jet, as is the case in the non-magnetic flow.

In the present paper, we have studied the effects of an axial magnetic field, which varies inversely as the radius vector, on the velocity distributions in a swirling jet of a viscous incompressible electrically conducting fluid which originates from a circular

slit. A perturbation on the Loitsianski model is applied and first order perturbation solutions for the velocity distributions are obtained. It is observed that the radial as well as tangential velocity decrease near the slit of the jet with the increase in the value of the magnetic-interaction parameter. These decelerated fluid particles, because of the law of conservation of mass, move in the positive axial direction and the points where they exactly balance the motion of incoming fluid have been calculated for different values of the magnetic-interaction parameter.

2. Formulation of the Problem

Let a rotating incompresible, viscous, electrically conducting fluid be discharged through a circular slit formed by two circular discs of negligible radii and negligible distance apart in the presence of variable axial magnetic field $\vec{B_z}$ (0, 0, $B_0 r^{-1}$) and mix with the same surrounding fluid being initially at rest (Fig. 1.). Taking the origin in

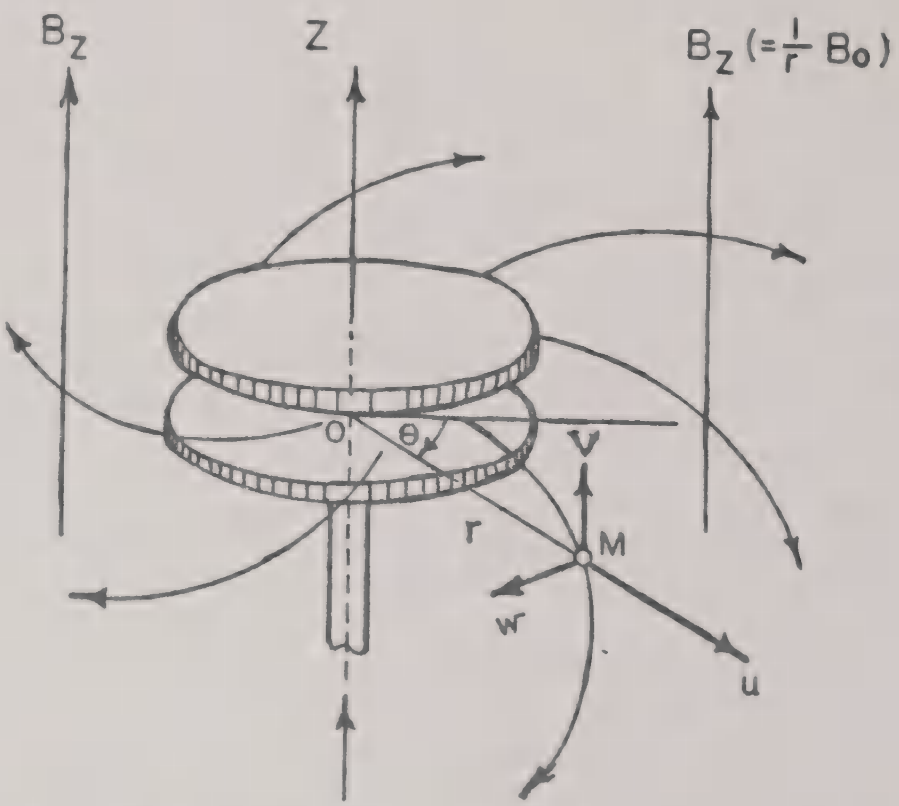

Fig. 1. Spread of MHD swirling jet which originates from a circular slit.

the slit, the boundary layer equations in cylindrical polar coordinates governing the fluid motion may be written as :

$$u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} - \frac{w^2}{r} = \nu \frac{\partial^2 u}{\partial z^2} - \frac{mu}{r^2} \qquad \ldots(2.1)$$

$$u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial z} + \frac{uw}{r} = \nu \frac{\partial^2 w}{\partial z^2} - \frac{mw}{r^2} \qquad ...(2.2)$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rv) = 0 \qquad ...(2.3)$$

where

$$m = \frac{\sigma_e B_0^2}{\rho}. \qquad ...(2.4)$$

The boundary conditions are

$$z = 0 : \frac{\partial u}{\partial z} = 0, v = 0, \frac{\partial w}{\partial z} = 0$$

$$z \to \pm \infty : u \to \frac{m}{r}, w \to \frac{m}{r^2}, \qquad ...(2.5)$$

where compatibility conditions are used in obtaining the conditions at infinity.

Besides these boundary conditions the following integral conditions, which govern the linear momentum flux (at a large radial distance from the slit) and the initial angular momentum flux of the jet respectively, should also be satisfied (Loitsianski[15]).

$$\lim_{r \to \infty} 2\pi\rho r \int_{-\infty}^{\infty} u_0^2 \, dz = J_0 \qquad ...(2.6)$$

$$2\pi \rho r^2 \int_{-\infty}^{\infty} u_0 w_0 \, dz = L_0 \qquad ...(2.7)$$

where u_0, v_0 and w_0 are the velocity distributions when the magnetic field is zero ($m = 0$) and are given by

$$u_0 = \frac{1}{r} \frac{\partial \psi_0}{\partial z}, v_0 = - \frac{1}{r} \frac{\partial \psi_0}{\partial r} \qquad ...(2.8)$$

$$\psi_0 = \alpha \sqrt{\nu} \sum_{i=0}^{\infty} r^{1-i} f_{1i}(\xi) \qquad ...(2.9)$$

$$w_0 = \beta \alpha^2 \sum_{i=1}^{\infty} r^{-i} g_{1i}(\xi) \qquad ...(2.10)$$

$$\xi = \frac{\alpha z}{n\sqrt{\nu}} \qquad ...(2.11)$$

$$\alpha = \left(\frac{3J_0}{4\pi \rho\sqrt{\nu}} \right)^{1/3}, \qquad ...(2.12)$$

$$\beta = \left(\frac{L_0}{J_0}\right)^{1/2} \qquad ...(2.13)$$

$$f_{10} = \tanh(\xi/2), \ f_{11} = 0.$$

$$g_{11} = 0, g_{12} = \beta/2 \operatorname{sech}^2(\xi/2). \qquad ...(2.14)$$

3. Analysis

The equation of continuity (2.3) is identically satisfied if we consider the Stoke's stream function ψ_m, such that

$$ru = \frac{\partial\psi_m}{\partial z}, rv = -\frac{\partial\psi_m}{\partial r}. \qquad ...(3.1)$$

The momentum equations (2.1) and (2.2) can be reduced to a set of ordinary differential equations if we consider the following series expansions for ψ_m and w, which are perturbations on Loitsianski's model[15]; i. e. equations (2.9) and (2.10)

$$\psi_m = \alpha\sqrt{\nu}\left[\sum_{i=0}^{\infty} r^{1-i} f_{1i}(\xi) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} r^{1-i}\left(\frac{m}{\alpha^2}\right)^j F_{ji}(\xi)\right] \qquad ...(3.2)$$

and

$$w = \beta\alpha^2\left[\sum_{i=1}^{\infty} r^{-i} g_{1i}(\xi) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r^{-i}\left(\frac{m}{\alpha^2}\right)^j G_{ji}(\xi)\right] \qquad ...(3.3)$$

where F_{ji} and G_{ji} are unknown functions of ξ to be determined.

Using (3.1) the expressions for u, v and w from (3.2) and (3.3) are obtained as

$$u = \frac{\alpha^2}{r}\left[\left(f'_{10} + \frac{1}{r} f'_{11} + ...\right) + \frac{m}{\alpha^2}\left(F'_{10} + \frac{1}{r}F'_{11} + ...\right) + \quad\right] \qquad ...(3.4)$$

$$v = \frac{\alpha\sqrt{\nu}}{r}\left[\left(\xi f'_{10} - f_{10}\right) + \frac{\xi}{r} f'_{11} + ... + \frac{m}{\alpha^2}\{(\xi F'_{10} - F_{10}) + \frac{\xi}{r} F'_{11} + ...\} + ...\right] \qquad ...(3.5)$$

and

$$w = \frac{\beta\alpha^2}{r}\left[(g_{11} + \frac{1}{r} g_{12} + ...) + \frac{m}{\alpha^2}(G_{11} + \frac{1}{r} G_{12} + ...\right] \qquad ...(3.6)$$

where a prime denotes differentiation with respect to ξ.

Substituting (3.4) to (3.6) in equations (2.1) and (2.2) and equating the coefficients of po wer of (m/α^2) and then of r^{-1},, we get the following set of ordinary differential equations for firstord er perturbation :

$$F'''_{10} + f_{10} F''_{10} + 2f'_{10} + F''_{10} f_{10} = F'_{10} \qquad \ldots(3.7)$$

$$F'''_{11} + f_{10} F''_{11} + 3f'_{10} F'_{11} = -2 (g_{12} G_{11} + G_{11} G^{12}), \qquad \ldots(3.8)$$

$$G''_{11} + f_{10} G'_{11} = 0, \qquad \ldots(3.9)$$

$$G''_{12} + f_{10} G'_{12} + f'_{10} G_{12} = g_{12} (1 - F'_{10}) - g'_{12} F_{10}. \qquad \ldots(3.10)$$

The corresponding boundary conditions from (2.5) are

$$\xi = 0 : F_{10} = 0,\ F''_{10} = 0;\ F_{11} = 0,\ F''_{11} = 0;\ G'_{11} = 0,\ G'_{21} \qquad \ldots(3.11)$$

$$\xi = \infty:\ F'_{10} = 1:\ F'_{11} = 0;\ G_{11} = 0;\ G_{12} = \beta. \qquad \ldots(3\text{-}12)$$

From the boundary conditions it follows that the solutions of equations (3.8) and (3.9) are

$$F_{11} = 0,\ G_{11} = 0 \qquad \ldots(3.13)$$

whereas the solution of eqn (3.10) may be obtained as

$$G_{12} (\xi) = \beta\ F'_{10} (\xi) \qquad \ldots(3.14)$$

Hence, we have to integrate only eqn. (3.7), under the boundary conditions (3.11) and (3.12). It may be noted that it is an exact differential equation and integrating it we get

$$(1 + \cosh \xi) F_{10} = C (\xi + \sinh \xi) + \int_0^{\xi} (1 + \cosh \xi) \log (1 + \cosh \xi)\, d\xi \qquad \ldots(3.15)$$

where the constant C is obtained by the boundary condition $F'_{10} (\infty) = 1$.

The integral in the R. H. S. of (3.15) can not be obtained easily in a compact form and therefore should be evaluated numerically for different values of ξ. Instead of this we have preferred to integrate the inhomogeneous third order ordinary linear differential equation (3.7) numerically by standard techniques using Runge-Kutta Gill method. It is found that

$$F'_{10} (0) = -2.52406 \qquad \ldots(3.16)$$

and the results are tabulated in Table 1.

TABLE I

ξ	f_{10}	f'_{10}	f''_{10}	F_{10}	F'_{10}	F''_{10}
0.0	0.00000	0.50000	0.00000	0.00000	−2.52406	0.00000
0.2	0.09967	0.49503	−0.04934	−0.50080	−2.46416	0.59318
0.4	0.19738	0.48052	−0.09484	−0.97310	−2.29127	1.11962
0.6	0.29131	0.45737	−0.13330	−1.41102	−2.02433	1.52667
0.8	0.37995	0.42782	−0.16255	−1.78337	−1.69056	1.78524
1.0	0.46212	0.39322	−0.18172	−2.08483	−1.32039	1.89209
1.5	0.63515	0.29829	−0.18946	−2.51253	−0.41160	1.64611
2.0	0.76159	0.20999	−0.15993	−2.53508	0.27420	1.08509
2.5	0.84828	0.14021	−0.11894	−2.28474	0.68553	0.58700
3.0	0.90515	0.09035	−0.08178	−1.88374	0.89189	0.26806
3.5	0.94138	0.05691	−0.05357	−1.41257	0.87891	0.10024
3.8	0.95624	0.04281	−0.04093	−1.11529	1.00030	0.04738
4.0	0.96403	0.03532	−0.03405	—	—	—
4.5	6.97803	0.02173	−0.02125	—	—	—
5.0	0.98661	0.01329	−0.01312	—	—	—
5.5	0.99186	0.00811	−0.00804	—	—	—
6.0	0.99506	0.00493	−0.00491	—	—	—

Finally, if we confine ourselves to the first two terms in the series expansions, we find

$$\frac{u}{(u_0)_{\xi=0}} = 2\left[f'_{10}\ (\xi) + \frac{m}{\alpha^2} F'_{10}\ (\xi)\right] \qquad \ldots(3.17)$$

$$\frac{v}{(\alpha\sqrt{\nu/r})} = (\xi f'_{10} - f_{10}) + \frac{m}{\alpha^2}\ (\xi F'_{10} - F_{10}) \qquad \ldots(3.18)$$

and

$$\frac{w}{(w_0)_{\xi=0}} = \frac{2}{\beta}\left[g_{12}\ (\xi) + \frac{m}{\alpha^2}\ G_{12}\ (\xi)\right] = \frac{u}{(u_0)_{\xi=0}}. \qquad \ldots(3.19)$$

The volume flux in the radial direction is given by

$$Q = 2\pi r \int_{-\infty}^{\infty} udz = 4\pi\alpha r\sqrt{\nu}\,[f_{10}\ (\infty) + \frac{m}{\alpha^2}\ F_{10})\ (\infty)]$$

$$= 4\pi\alpha r\sqrt{\nu}\,[1 - 1.11529\ \frac{m}{\alpha^2}]. \qquad \ldots(3.20)$$

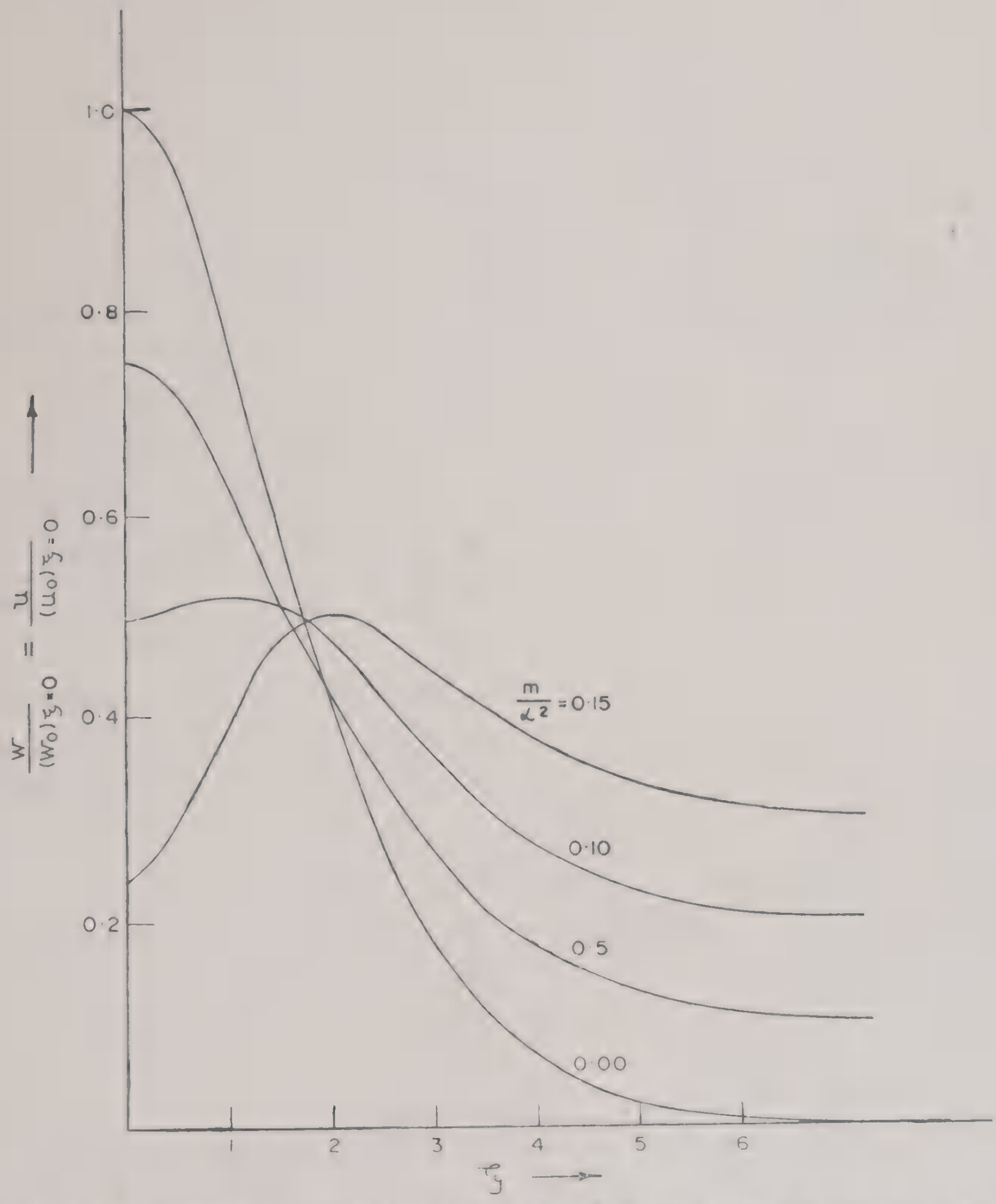

FIG. 2. Radial and transverse velocity distribution of a MHD swirling jet which originates from a circular slit.

This shows that the volume flux in the radial direction decreases linearly with respect to $m\alpha/l^2$.

The points where the incoming flow in the axial direction will be exactly balanced by outward flow of the decelerated fluid particles is obtained from (3.18) when $v = 0$ i. e.

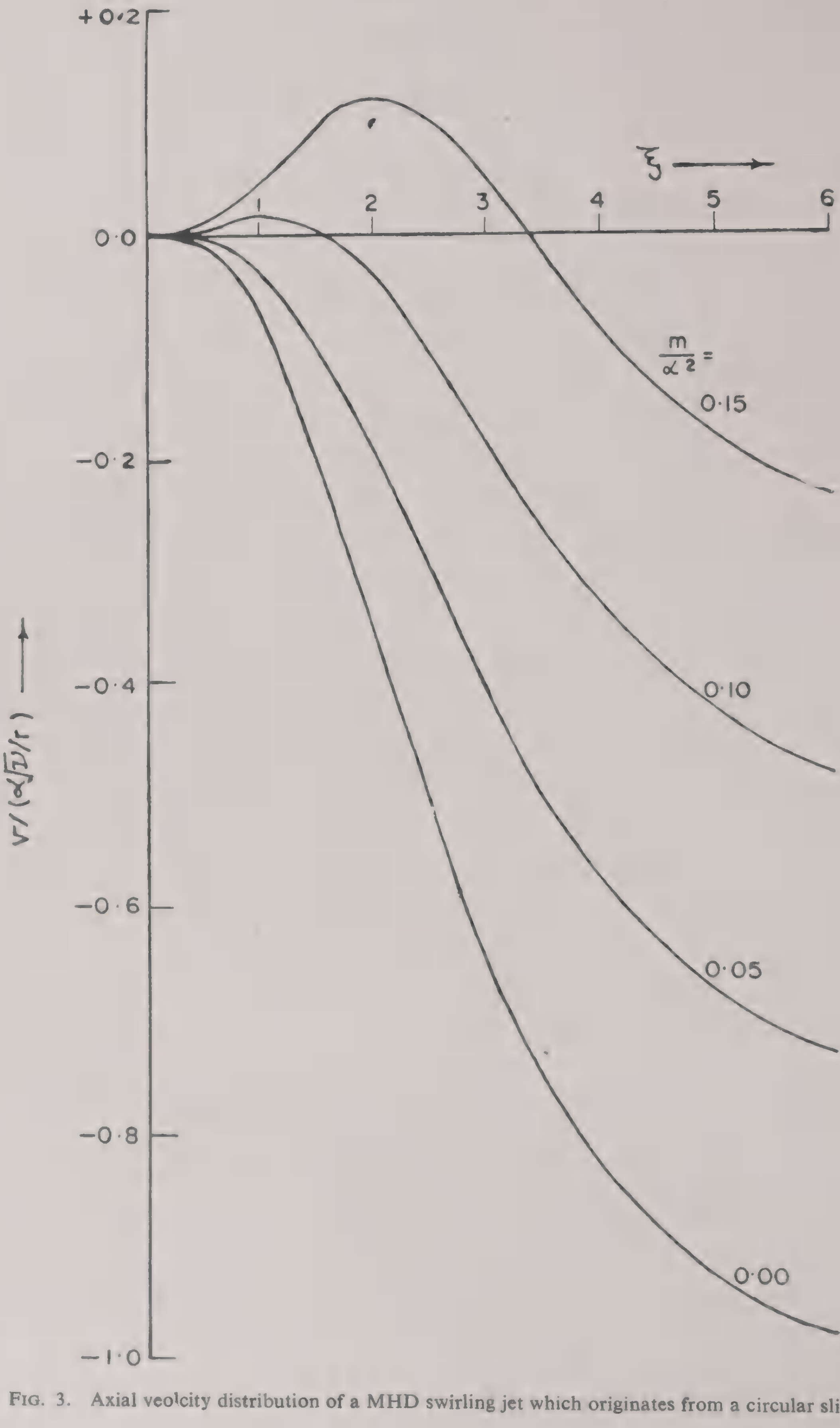

FIG. 3. Axial veolcity distribution of a MHD swirling jet which originates from a circular slit.

$$\frac{m}{\alpha^2} = \frac{f_{10}(\xi) - \xi f'_{10}(\xi)}{\xi F'_{10} - F_{10}(\xi)} . \qquad \ldots(3.21)$$

4. Numerical Discussion

As the magnetic field can only retard and not produce flow in the reverse direction, the velocity field and the volume flux should be non-negative. Therefore the range of the magnetic-interaction parameter in the present series solution should be $0 \leqslant m/\alpha^2 < 0.198$, a range in which the fluid has always a forward flow.

The non-dimensional velocity profiles for radial and transverse velocity components, for various values of the magnetic interaction parameter (m/α^2), are plotted against the similarity variable in Fig. 2 and for the axial velocity in Fig. 3 respectively. The effect the magnetic field is to decrease the radial as well as tangential velocity near the slit of the jet and to increase it far from the slit. From Fig. 3, it is clear that the axial velocity increases algebraically throughout the velocity field with the increase in magnetic-interaction parameter.

Following physical explanation may be given for the results obtained :

Due to the action of the centrifugal forces the fluid near the slit of the jet will be thrown outward and to compensate this a flow in axial direction towards the slit will follow. Now, the Lorentz force retards the radial and tangential motion of the fluid and therefore these decelerated fluid particles, from the law of conservation of mass, start moving in the positive axial direction and thus affect the incoming axial velocity, which moy eventually change its direction. It may happen that for a prescribed value of m/α^2 the incoming flow is exactly balanced by the upward flow. Such points may be obtained from Fig. 3, where v is zero.

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References

1. H. Schlichting, *ZAMM* 13 (1933), 260.
2. W. Bickley, *Phil. Mag.* 7 (1939), 727.
3. R. L. Peskin, *Phys. Fluids* 6 (1963). 643.
4. D. C. Smith and A. B. Cambel, *Phys. Fluids* 8 (1965), 2107.
5. A. Pozzi and A. Bianchini, *ZAMM* 52 (1972), 523.
6. J. L. Bansal *ZAMM* 55 (1975), 479.
7. M. L. Gupta, Ph. D. Thesis, University of Jodhpur, Jodhpur, 1980.
8. J. J. Mishra and J. L. Bansal, Magnetogasdynamic circular jet in a radial magnetic field. *IL Nuovo Cimento*, 86B (1985), 39.

9. L. G. Loitsianski, *Prikl. Mekh.* 17 (1953), 9.

10. H. Görtler, *Rev. Math. Hispani Am.* **14** (1954), 143.

11. M. H. Steiger and M. H. Bloom, *Axially symmetric laminar free mixing with swirl, Proceedings of the Heat transfer and Fluid Mechanics, Institute, Stanford University Press, Stanford, Calif.*, 1961.

12. A. J. Reynolds, *JFM* 14 (1962), 241.

13. A. Chervinsky, *AIAA* 6 (1968), 912.

14. J. J. Mishra and J. L. Bansal, *Proo. Indian Acad. Sc.* (India) 1986 (Communicated)

15. L. G. Loitsianski, *Laminare Grenzschichten*. Akademie-Verlag. Berlin, 1967.

Eratta

"Corrections to The neighbourhood number of a graph"
by P. P. Kale and N. V. Deshpande, *Indian Journal of pure appl. Math.* 19 (1988), 927–929

It needs correction as the result $n_0 \leqslant 2n_0'$ is false. The last proposition and its proof should read as follows.

Proposition : For any graph G,

$$n_0 \leqslant 2\gamma'. \qquad \ldots(6)$$

Proof : Let $T = \{x_1, x_2, \ldots, x_k\}$ be a line dominating set of minimum cardinality and $x_i = u_i v_i$, $1 \leqslant i \leqslant k$. Let $H = \bigcup_{i=1}^{k} \{u_i, v_i\}$. As every line in G is incident with a point in H, H is a neighbourhood set of G, Hence $n_0 \leqslant | H | \leqslant 2 | T | = 2\gamma'$. This proves the proposition.

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